A WZ PROOF OF A "CURIOUS" IDENTITY

Shalosh B. Ekhad and Mohamud Mohammed

Abstract: We give a short WZ-proof of an identity that appeared recently in Integers.

In [1] and [2], the identity,

$$\sum_{i=0}^{m} (x + m + 1)(-1)^i \binom{x+y+i}{m-i} \binom{y+2i}{i} - \sum_{i=0}^{m} \binom{x+i}{m-i} (-4)^i = (x - m) \binom{x}{m} \quad \text{..........(*)}$$

was proved using generating functions and double recursions respectively.

Here we cleverly construct the function

$$g(i) = -\binom{m}{i} \frac{(x^2 - 2ix - x + i + i^2)}{2(1+i)(x+i+2)(x+i+1)(-2)^i}$$

with the motives that

$$\sum_{i=0}^{m} \binom{x+i}{m-i} (-4)^i = (-2)^m (x + m + 1) \left( \frac{1}{x+1} + \sum_{i=0}^{m-1} g(i) \right) \quad \text{..........(**)}$$

The first sum, call it $T(m)$, on the LHS of (*) divided by $(x+m+1)$ satisfies the following recurrence equation as found by EKHAD.

$$a_0(m)T(m) + a_1(m)T(m+1) + a_2(m)T(m+2) + a_3(m)T(m+3) = 0,$$

where

$$a_0(m) = 2(x - m - 1)(x - m - 2); a_1(m) = -(x - m - 2)(2y - x + 5m + 11),$$

$$a_2(m) = (-yx + 3ym - 2xm + 4m^2 + 8y - 5x + 21m + 28); a_3(m) = (m + 3)(y + m + 3)$$

Now, it is routine(use maple!) to check that the sum of the RHS's of (*) and (**) divided by $(x + m + 1)$ satisfies the recurrence equation. Moreover, both sides agree for $m = 0, 1, 2$. Q.E.D.

REFERENCES


1 Department of Mathematics, Rutgers University, Piscataway, NJ 08854, USA. mohamud@math.rutgers.edu