Computing the Generating Function of a C-finite sequence

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Input: A C-finite sequence $a(n)$ that is a solution of a (homogeneous) **linear recurrence equation with constnt coefficient**

$$a(n) = R_1 a(n-1) + \ldots + R_d a(n-d) \quad ,$$

subject to the initial conditions

$$a(0) = a_0, \ldots, a(d-1) = a_{d-1} \quad .$$

Note: this 'infinite' sequence is a **finite** object, and we represent it in class as a pair of lists `[INI,REC]` where

$$INI = [a_0, \ldots, a_{d-1}] \quad , \quad REC = [R_1, \ldots, R_d] \quad .$$

The (ordinary) generating function of a sequence $a(n), 0 \leq n < \infty$ is, by definition

$$f(x) = \sum_{n=0}^{\infty} a(n) x^n \quad .$$

How to go from the C-finite description to the generating function

For $n \geq d$ we have

$$a(n) = R_1 a(n-1) + \ldots + R_d a(n-d) \quad ,$$

Hence

$$f(x) = \sum_{n=0}^{\infty} a(n) x^n = \sum_{n=0}^{d-1} a(n) x^n + \sum_{n=d}^{\infty} a(n) x^n$$

$$= \sum_{n=0}^{d-1} a(n) x^n + \sum_{n=d}^{\infty} \left(R_1 a(n-1) + R_2 a(n-2) + \ldots + R_d a(n-d)\right) x^n \quad .$$

Distributing, we have

$$f(x) = \sum_{n=0}^{d-1} a(n) x^n$$

$$+ \sum_{n=d}^{\infty} R_1 a(n-1) x^n + \sum_{n=d}^{\infty} R_2 a(n-2) x^n + \ldots + \sum_{n=d}^{\infty} R_d a(n-d) x^n \quad .$$

Rewritng: we get

$$f(x) = \sum_{n=0}^{d-1} a(n) x^n$$

$$+ R_1 x \sum_{n=d}^{\infty} a(n-1)x^{n-1} + \; R_2 x^2 \sum_{n=d}^{\infty} a(n-2)x^{n-2} + \ldots + \; R_d x^d \sum_{n=d}^{\infty} a(n-d)x^{n-d} \quad .$$

Changing the index of summation we get

$$f(x) = \sum_{n=0}^{d-1} a(n)x^n$$

$$+ R_1 x \sum_{n=d-1}^{\infty} a(n)x^n + \; R_2 x^2 \sum_{n=d-2}^{\infty} a(n)x^n + \ldots + \; R_d x^d \sum_{n=0}^{\infty} a(n)x^n \quad .$$

It follows that

$$f(x) = POLYNOMIAL_{d-1}(x) + (R_1 x + dots + R_d x^d) f(x)$$

where $POLYNOMIAL_{d-1}(x)$ is **some** polynomial of degree $d-1$. Hence

$$f(x)(1 - (R_1 x + dots + R_d x^d)) = POLYNOMIAL_{d-1}(x) \quad ,$$

and hence

$$f(x) = \frac{POLYNOMIAL_{d-1}(x)}{)(1 - (R_1 x + dots + R_d x^d))} \quad .$$

This explains why $f(x)$ is always a rational function, why its denominator is what it is. As for the numerator, the best way is to find it empirically as we did in class.

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