TEST YOUR C.Q. [Conjecture-Quotient]

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PREFACE. According to Gian-Carlo Rota, Richard Feynman's secret to being a genius is to be aware of many open problems, and everytime you learn a new trick, try it out on all them. Most of the time, the new trick won't do you any good, but once or twice in a lifetime, it would exactly solve one of the problems, and then everybody would say: Gee, what a genius!

Hence according to Feynman (according to Rota), a necessary condition for being a genius is knowing about open problems. According to Zeilberger, it is also important to know about ex-conjectures, and about how they were solved [this will teach you some neat tricks]. Hence the following trivia quiz is highly non-trivial!

Some of the conjectures are extremely famous (like **15**), some are fairly famous (like **13** and **14**), but some are really obscure (like **6** and **22**). Good luck!

INSTRUCTIONS. To each of the statements in section I, match the mathematician in section II [if any] who conjectured it, the mathematician in section III, [if any], who believed he had a proof, but, alas, was mistaken, and the mathematician(s) in section IV[if any] who first settled the conjecture, [so far] for good. Also state whether the statement is true, false, undecidable, or still open.

Send solutions to `zeilberg@math.temple.edu`. The top three scorers [if any] will receive each a package of M&Ms.

I. The CONJECTURES

1. You can square a circle.

2. You can duplicate a cube and trisect an angle.

3. Every even integer is the sum of two primes.

4. Every integer can be written as a sum of k^{th} powers (for any k).

5. $2^{2^n}+1$ is always prime.

6. $x^n + y^n = z^n$ and $n > 2$ imply that $xyz = 0$.

7. There do not exist non-zero integers A, B, C, D such that $A^4 + B^4 + C^4 = D^4$.

8. There do not exist Greek-Latin squares of order $2 + 4k$ for all $k \geq 2$.

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9. Every positive-definite real polynomial, in any number of variables, can be written as a sum of squares of rational functions.

10. A simply connected $n-$dimensional closed manifold is homeomorphic to an $n-$sphere.

11. Analytic Index=Topological Index.

12. If the Jacobian determinant of a polynomial transformation from n-space to $n-$space is identically [a non-zero] constant, then the inverse mapping is also polynomial.

13. The number of $n \times n$ Alternating Sign Matrices is $[1!4!7! \ldots (3n-2)!]/[n!(n+1)! \ldots (2n-1)!]$.

14. The coefficient of $x_1^0 \ldots x_n^0$ in the Laurent polynomial

$$\prod_{1 \le i < j \le n} \{\prod_{r=0}^{a_i}(1 - q^r x_i/x_j) \prod_{r=1}^{a_j+1}(1 - q^r x_j/x_i)\}$$

equals $[a_1 + \ldots a_n]!/([a_1]! \ldots [a_n]!)$, where $[m]! = (1)(1+q)(1+q+q^2)\ldots(1+q+\ldots q^{m-1})$.

15. For every integer $k \ge 0$, the Taylor coefficients of $(1-q)^k/[(1-q)(1-q^2)\ldots(1-q^k)]$ at $q=0$ alternate in sign (weakly, i.e. 0 can be taken to be positive or negative).

16. $\sum_{n=1}^{\infty} n^{-3}$ is irrational.

17. Every set in R^d is a union of $d+1$ sets of smaller diameter.

18. You can hear the shape of a drum (i.e. the spectrum of the Dirichlet problem on a plane region determines its shape).

19. Any univalent function f on the unit disc such that $f(0) = 0$ and $f'(0) = 1$ has $|f^{(n)}(0)/n!| \le n$.

20. Every planar map is 4-colorable.

21. There is nothing between $\aleph_0$ and the cardinality of R.

22. All non-trivial zeros of the meromorphic continuation of $\sum_{n=1}^{\infty} n^{-z}$ have $Rez = 1/2$.

23. Every finitely generated projective module over $k[t_1, \ldots, t_n]$ is free.

24. There is a decision-procedure to decide whether any given diophantine equation has solutions.

25. Let $\tau(n)$ be the coefficient of q^n in $q(1-q)^{24}(1-q^2)^{24}(1-q^3)^{24}\ldots$ then $|\tau(n)| \le d(n)n^{11/2}$, where $d(n)$ is the number of divisors of n.

26. The analog of **22** over finite fields.

27. For any root system R the constant term of $\prod_{\alpha \in R}(1 - x^\alpha)^a$ is $\prod_{i=1}^{l}\binom{d_i a}{a}$, where $d_1, \ldots, d_l$ are the *fundamental invariants* of R.

28. If the section function of a centered convex body in E^n, $n \geq 4$ is smaller than that of another such body, then its volume is also smaller.

29. The class of NP functions *strictly* contains the class of P functions.

30. A finite group of odd order is solvable.

31. Every differential operator with constant coefficients has a fundamental solution.

32. The general solution of an arbitrary system of linear partial differential equations with constant coefficients can be expressed as an ' infinite linear combination' of exponential-polynomial solution (expressed as integrals over multiplicity varieties).

33. The complement of a perfect graph is perfect.

34. The permanent of an $n \times n$ doubly-stochastic matrix is $\geq n!/n^n$.

35. $2^{\sqrt{2}}$ is transcendental.

36. The number of totally symmetric plane partitions with largest part $\leq n$ is equal to

$$\prod_{1 \leq i \leq j \leq k \leq n} \frac{i+j+k-1}{i+j+k-2} \quad .$$

37. The number of 2-stack-sortable permutations on n objects is $2(3n)!/((2n+1)!(n+1)!)$.

38. The number of rational points of an algebraic curve C defined over Q with genus larger than 1 is finite.

39. The eignevalues of the Hecke operators for Mass forms of Galois type are algebraic.

40. Euler's constant $\gamma := \lim_{n \to \infty} [(\sum_{i=1}^{n} 1/i) - \log n]$ is irrational.

41. No packing of spheres in Euclidean $3-$space has density exceeding that of the face-centered cubic lattice packing, which is $\pi/\sqrt{18} = 0.74048\cdots$.

42. $\beta := \lim_{n \to \infty} 2nE_{2n}(|x|)$, (where $E_{2n}(f)$ is the error in the best $L^\infty[-1,1]$ approximation of f by polynomials of degree $\leq 2n$) is equal to $\frac{1}{2\sqrt{\pi}}$.

II. The CONJECTURERS

George Andrews; Anon.[or I don't know] (6); Claude Berge; Stephen Bernstein; Ludwig Bieberbach; Karol Borsuk; William S. Burnside; Busemann-Petty; Lehonard Euler(2); Pierre Fermat(2) ; Christian Goldbach; Frederick Guthrie; David Hilbert(5); Marc Kac; Richard Karp(?); Keller; Johann Kepler ; Ian G. Macdonald; Robert Mills-David Robbins-Howard Rumsey; Louis J. Mordell; Andrew Odlyzko; Oracle of Delphi(2); Henry J. Poincaré; Srinivasa Ramanujan; Bernard Riemann;

Jean-Pierre Serre; Richard Stanley; Bartel L. van der Waerden; Edward Waring; André Weil; Julian West.

III. The WOULD-BE PROVERS

Paul Appel [published in Comptes Rendus, 1926]; D.Blasius-Laurent Clozel-D.Ramakrishnan [published in Comptes Rendus 1989]; Louis de Branges[published in J. Math. Anal. Appl., 1972]; Louis de Branges[unpublished but announced several times], Hans Rademacher [announced, submitted but retracted], and many others; David Hilbert [published in Annalen]; Wu-Yi Hsiang [published in Inter. J. Math, 1993]; A.B. Kempe [published in Amer. J. Math., 1879]; Ernst E. Kummer[submitted], Gabriel Lamé[published in Comptes Rendus], C. L. Ferdinand Lindemann [published in Annalen]; McNeish[1927, published in Annals of Math, claimed (incorrectly)that conj. is true]; Ed Nelson [announced, but not published]; Gaoyong Zhang[published in Annals of Math., 1994].

IV. The PROVERS and DISPROVERS

Roger Apéry; Ken Appel-Wolfgang Haken; Emil Artin; Michael Atiyah- Isadore Singer; Louis de Branges; Paul Cohen; Piere Deligne(2); G.P. Egoritsjev-D.I. Falikman; Leon Ehrenpreis- Bernard Malgrange; Leon Ehrenpreis- Viktor Palamadov; Noam Elkies; Lehonard Euler; Gerd Faltings; Walter Feit and John G. Thompson; Evariste Galois[essentially]; Aleksander O. Gelfond-Theodor Schneider; Carolyn Gordon and David Webb; David Hilbert; Jeff Kahn and Gil Kalai; C.L. Ferdinand Lindemann; Lászo Lovász; Yuri Matiyachevich; John Milnor (n >4)- Simon Donaldson, Michael Freedman ($n = 4$)- still open($n = 3$); Eric Opdam [after ingenious proofs of the G_2 case by Laurent Habsieger and Doron Zeilberger]; Danile G. Quillen-Mihail J. Suslin; Dennis Stanton and Doron Zeilberger; Still Open [as far as I know](8); John Stembridge; R.C. Bose and D.K.Ray-Chaudhuri; Richard S. Varga and Amos Carpenter; Andrew Wiles; Doron Zeilberger(2); Doron Zeilberger and Dave Bressoud.

CONCLUSION.

I will let you decide for yourself whether you are a CG (Conjecture-Genius), or CR (Conjecture-Retard), or somewhere in between. Also note the distinguished set of mathematicians who made it to section III (many of whom also show up in section IV), so learn from Gian-Carlo Rota who said not to worry about your mistakes.