

Five More Proofs of the Cosine Addition Formula (Inspired by Mark Levi's Perpetuum Mobile Proof)

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Dedicated to my hero Mark Levi

Every time I get *SIAM News*, I immediately turn to Mark Levi's wonderful column *Mathematical Curiosities* <https://www.marklevimath.com/sinews>. In the current issue, Levi [L] gave a *proof from the book* of the trigonometric identity $\cos(\alpha + \beta) = \cos \beta \cdot \cos \alpha - \sin \beta \cdot \sin \alpha$, by showing that it follows from the non-existence of **Perpetual Motion**. This reminded me of five other proofs, none of them, admittedly, as nice as Mark Levi's proof (although the last one is a close second).

Pre-calculus: See the Wikipedia article *List of Trigonometric Identities*. \square

Linear Algebra: Rotating the vector $[1, 0]^T$ by an angle of α gives the vector $[\cos \alpha, \sin \alpha]^T$. Rotating the vector $[0, 1]^T$ by an angle of α gives the vector $[-\sin \alpha, \cos \alpha]^T$. By **linearity**, the **rotation matrix**, R_α is

$$R_\alpha = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} .$$

Now use **matrix multiplication** and the fact that $R_{\alpha+\beta} = R_\alpha R_\beta$. \square

Complex Variable: Take the real parts of both sides of $e^{i(\alpha+\beta)} = e^{i\alpha} e^{i\beta}$ \square

Differential Equations: Both sides satisfy the differential equation (viewed as a function of α) $y'' + y = 0$, subject to the initial conditions $y(0) = \cos \beta$, $y'(0) = -\sin \beta$. Now use uniqueness. \square

Combinatorics: $\cos \alpha$ (resp. $\sin \alpha$) is the **exponential generating function of increasing sequences of integers of even length** (resp. of **odd length**) with weight $(-1)^{\text{length}/2}$ (resp. $(-1)^{(\text{length}-1)/2}$), see [Z]. Hence $\cos(\alpha + \beta)$ is the exponential generating function of two-colored increasing sequences of even length, with say, colors α and β . If the number of integers colored α is even (resp. odd) we get the first term (resp. second term) on the right. \square

References

[L] Mark Levi, *Cosine Addition Formula and Perpetual Motion*, *SIAM News* **55 #3** (April 2022), p.7.

[Z] Doron Zeilberger, *Enumerative and Algebraic Combinatorics*, in: "*Princeton Companion to Mathematics*" , (Timothy Gowers, ed.), Princeton University Press, 2008, 550-561.
<https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/enuPCM.pdf> .

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