**Chapter III: Games of Pure Chance Generated by Gambler's Ruin with Unlimited Credit: The Fuss-Catalan case**

In Chapter II, we only allowed positive steps. Now we will also allow negative steps, and treat games that may be viewed as a "gambler's ruin with infinite credit", with *arbitrary* 'die'. In the next chapter we will treat the case of a *general* die, while in this chapter we only consider two-faced dice, where one of the faces is marked 1 and the other markded $-k$, and the more difficult case where one of the faces is marked $-1$ and the other marked $k$. We will start with their intersection, the very classical case of $\{-1, 1\}$, treated in Feller's classic [F]. However, even in this case we will be able to go beyond Feller, since he did not use a computer.

The general set-up, to be considered in full generality in Chapter IV is as follows.

On the discrete line, you start at the origin $x = 0$, and there is a fixed allowed set of steps consisting of both positive and negative integers and a probability distribution on them, let's call it $\mathcal{P}$. You are allowed to go as far left as possible (i.e. you can owe as much as necessary). At each round, you roll the $\mathcal{P}$ die, and move accordingly. You win as soon as you reach a location $\geq 1$, or more generally when you reach a location $\geq n$. In other words, your goal is to exit the casino with at least one dollar (or more generally, at least $n$ dollars). In the two-player (or multi-player) version, the players take turns rolling the $\mathcal{P}$ die, and whoever achieves the goal first is declared the winner. As before, we will first discuss the solitaire game, where the goal is to reach it as soon as possible.

**Classical Gambling: Winning a dollar or losing a dollar**

Let's start with the simplest, most classical case, of simple random walk, where you start with 0 dollars, and at each round you win a dollar with probability $p$ and lose a dollar with probability $1 - p$. The expected gain at each individual round is $p \cdot 1 + (1 - p) \cdot (-1) = 2p - 1$, so if $p > \frac{1}{2}$, then sooner or later you will reach your goal. If $p < \frac{1}{2}$, then you may never make it, sliding down to infinite debt. In the border-line case of a **fair** coin, $p = \frac{1}{2}$, as we will soon see, you are also guaranteed to 'eventually' be in possession of 1 dollar (and more generally, $n$ dollars for each $n > 0$, as big as you wish). Alas, as we will also soon see, the expected time until that happens is infinite, and since life is finite, there is a good chance that when you will pass away, your heirs will have a huge debt.

**Analyzing Gambling histories**

For typographical clarity, let's denote $-1$ by $\bar{1}$.

Our **alphabet** is $\{-1, 1\}$=$\{\bar{1}, 1\}$. A 'gambling history' consists of a **word** that ends in 1, whose sum is 1, and whose proper partial sums are all non-positive. Obviously the length of such a game is odd.

If you are really lucky, you exit after one step, since you won a dollar right away.

If you lost a dollar at the first round, you can recover at the second round, and then win a dollar at the third round. Etc.

For the sake of clarity and concreteness, let's list the first few 'histories'.

Length 1: $\{1\}$. Probability $= p$.

Length 3: $\{\bar{1}\,1\,1\}$. Probability $= p^2\,(1-p)$.

Length 5: $\{\bar{1}\,1\,\bar{1}\,1\,1 \quad , \quad \bar{1}\,\bar{1}\,1\,1\,1\}$. Probability $2 \cdot p^3\,(1-p)^2$.

Length 7:

$$\{\bar{1}\,1\,\bar{1}\,1\,\bar{1}\,1\,1 \quad , \quad \bar{1}\,1\,\bar{1}\,\bar{1}\,1\,1\,1 \quad , \quad \bar{1}\,\bar{1}\,1\,1\,\bar{1}\,1\,1 \quad , \quad \bar{1}\,\bar{1}\,1\,\bar{1}\,1\,1\,1 \quad , \quad \bar{1}\,\bar{1}\,\bar{1}\,1\,1\,1\,1\} \quad ,$$

with probability $5 \cdot p^3\,(1-p)^2$.

It is useful, for humans, to visualize such a history as a lattice path in the discrete plane starting at $(0,0)$ where $\bar{1}$ corresponds to a step $(1,-1)$ and 1 corresponds to a step $(1,1)$. For example, the word (gambling history)

$$\bar{1}\,\bar{1}\,1\,1\,\bar{1}\,1\,1 \quad ,$$

corresponds to the walk

$$(0,0) \to (1,-1) \to (2,-2) \to (3,-1) \to (4,0) \to (5,-1) \to (6,0) \to (7,1) \quad .$$

Let's study the *anatomy* of such histories, or equivalently, paths . Obviously they are all of odd length, and they all end with 1. So we can write, for *any* history $W$

$$W = U\,1 \quad ,$$

where $U$ is a word that sums to 0, all whose partial sums are non-positive. Such words are called *Dyck words*.

Let's analyze such a Dyck word $U$ or rather its corresponding path from $(0,0)$ to $(2n,0)$, say. Of course, it may be the empty word, but if it is not, let $(2r,0)$ $0 < r \leq n$ be the **first** time that it hits the $x$-axis. Then we can write

$$U = U_1\,U_2 \quad ,$$

where $U_2$ is another word of that kind (of length $2n-2r$), but $U_1$, consisting of the first $2r$ letters of $U$, has the special property that all its partial sums (except the 0-th and the last) are **strictly negative**, or in terms of its path, except for its starting and ending points, they lie **strictly** below the $x$-axis. Such a word must **necessarily** start with a $\bar{1}$ and end with a 1, and may be written as $\bar{1}U_3 1$, where $U_3$ is an arbitrary Dyck word. Conversely, for any Dyck word $U_3$, $\bar{1}\,U_3\,1$ corresponds to such a 'strictly below the $x$-axis' path. So we have the (context-free) *grammar*

$$U = EmptyWord \vee \bar{1}\,U\,1U \quad , \qquad (DyckGrammar)$$

where now $U$ stands for 'an arbitrary Dyck word'.

let $x_1$ and $x_{-1}$ be **commuting** variables.

For any word $u = u_1 \dots u_m$, let the *weight* of $u$ be $x_{u_1} \cdots x_{u_m}$. For example,

$$weight(\bar{1}\,\bar{1}\,\bar{1}\,1\,1\,1\,\bar{1}\,1\,1) = x_{-1}x_{-1}x_{-1}x_1x_1x_1x_{-1}x_1x_1 \;=\; x_{-1}^4 x_1^5 \quad .$$

Let $F(x_{-1}, x_1)$ be the **weight enumerator** of the set of Dyck words, i.e. the *sum* of all the weights of all these words, a certain **formal power series** in $x_{-1}, x_1$.

Obviously the weight of the empty word is 1 (the empty product), hence applying *weight* to $(DyckGrammar)$, we get the **quadratic equation**

$$F = 1 \,+\, x_{-1}\, F\, x_1\, F \quad .$$

Abbreviating $X = x_{-1}\, x_1$, we get

$$F = 1 \,+\, X\, F^2 \quad .$$

Recalling what we learned in seventh grade (or what the Babylonians knew more than 3000 years ago), we can express $F$ **explicitly**

$$F = \frac{1-\sqrt{1-4X}}{2X} \quad .$$

Recalling what we learned in 12-th grade (or what Isaac Newton knew more that 300 years ago) we can write

$$F = \sum_{m=0}^{\infty} \frac{(2\,m)!}{m!\,(m+1)!} X^m \quad ,$$

implying the fact that the number of Dyck paths of length $2m$ is the super-famous **Catalan** number $C_m = \frac{(2m)!}{m!\,(m+1)!}$, that is the subject of Richard Stanley's modern classic [St], and the most popular sequence, `A108`, in the great OEIS [Sl].

The above is the standard, very boring proof of that famous fact. We know at least a dozen proofs, some of them are given in [St]. Here is one of our favorite proofs due to Aryeh Dvoretzky and Theodore Motzkin [DM].

The fact that the number of Dyck paths of length $2m$ equals the Catalan number $C_m$ is equivalent the fact that the number of words in $\{1, -1\}$ of length $2m+1$ whose sum is $1$ and all whose proper-partial sums are non-positive is $C_m$. Every word of length $2m+1$ in $\{-1, 1\}$ that adds up to $1$ has $m+1$ '1' and $m$ '$\bar{1}$'. There are $\binom{2\,m+1}{m}$ such words. The $2\,m\,+\,1$ *cyclic shifts* of each such word are all **different** (why?), and exactly one of them has the property that its partial sums are all non-positive (why?). Hence the number of gambling histories that we are interested in is $\frac{1}{2m+1} \cdot \binom{2m+1}{m} = C_m$.

**Enter Probability**

So far what we did was *enumerative combinatorics*. We found out that the weight-enumerator of the set of Dyck words is

$$\frac{1-\sqrt{1-4x_{-1}x_1}}{2\,x_{-1}x_1} \quad ,$$

and hence the weight enumerator of words in $\{-1,1\}$ that add-up to 1, and such that all their proper partial sums are $\leq 0$, is $x_1$ times that, i.e.

$$\frac{1-\sqrt{1-4x_{-1}x_1}}{2\,x_{-1}} \quad .$$

Assume that each round in the gambling game is **independent** of the other ones, and for each of them the probability of winning a dollar is $p$, and hence of losing a dollar is $1-p$. Plugging-in $x_{-1}=(1-p)\,t$, $x_1=p\,t$, in the above explicit enumerating generating function, we get the following human-generated, well-known (see [F]) proposition.

**Proposition 5**: The **probability generating function** of the random variable 'numer of rounds it takes until the first time you have one dollar', if you start with 0 dollars and at each round you win a dollar with probability $p$ and lose a dollar with probability $1-p$, let's call it $g(t)$ is

$$g(t)=\frac{1-\sqrt{1-4\,(1-p)\,pt^2}}{2\,(1-p)\,t} \quad .$$

So far all our power series were *formal*, but it is easy to see that if $p \geq \frac{1}{2}$ then plugging-in $t=1$ leads to a convergent series, that sums-up to 1, in agreement with the obvious fact that if $p>\frac{1}{2}$ sooner or later you will succeed, and the slightly less obvious fact that it is still true when $p=\frac{1}{2}$. If $p<\frac{1}{2}$, then we must take the other sign of the square-root, leading to the classical and well-known fact that the probability of one day having one dollar in your possession is $\frac{p}{1-p}$.

More generally, suppose that your goal in life is not just to exit the casino with one dollar, but you want to make $n$ dollars. Since each additional dollar is yet another 1-dollar game, we immediately get.

**Proposition 5'**: The **probability generating function** of the random variable 'numer of rounds it takes until the first time you have $n$ dollars', if you start with 0 dollars and at each round you win a dollar with probability $p$ and lose a dollar with probability $1-p$, is given by

$$\left(\frac{1-\sqrt{1-4\,(1-p)\,pt^2}}{2\,(1-p)\,t}\right)^n \quad .$$

From now let's assume that $p \geq \frac{1}{2}$. To get the **expected duration** we can sill do it by hand, find $(g(t)^n)' = ng(t)^{n-1}g'(t)$, then compute $g'(t)$, plug-in $t=1$ and simplify, getting that the expectation is $\frac{n}{2p-1}$.

For the $k$-th moment, we compute $(t\frac{d}{dt})^k(g(t)^n)$, plug-in $t=1$, and simplify, expressing all higher derivatives of $g(t)$ in terms of $g(t)$ and $t$, followed by substituting $t=1$.

An even better way, that would be the only way later on when we do the general gambling caes, is to use *implicit differentiation*, using the relation

$$f(t) \,=\, 1+p\,(1-p)\,t^2 f(t)^2 \quad ,$$

and its implied relation for $g(t) = p\,t\,f(t)$.

It turns out that if you use the explicit expression $g(t) = \frac{1-\sqrt{1-4\,(1-p)\,pt^2}}{2\,(1-p)\,t}$ all the radicals disappear, and if you use implicit differentiation, and then plug-in $t = 1$, you never have to divide 0 by 0, so either way you would get that all the moments are **polynomials** in $n$ and rational functions in $p$. In particular, if $p$ is a rational number, then they are all also rational numbers. The expectation, is $\frac{n}{2p-1}$.

For higher moments, We get the following computer-generated proposition.

**Proposition 6:** Let $X_{n,p}$ be the random variable "Number of rounds until reaching $n$ dollars for the first time" in a gambling game where the probability of winning a dollar is $p$ and of losing a dollar is $1-p$. Assume that $p > \frac{1}{2}$. We have

$$E[X_{n,p}] \,=\, \frac{n}{2p-1} \quad .$$

$$Var[X_{n,p}] \,=\, \frac{4\,n\,p\,(1-p)}{(2p-1)^3} \quad .$$

The *skewness* (aka scaled third moment about the mean) is

$$\alpha_3[X_{n,p}] \,=\, \left(-2\,p^2+2\,p+1\right)(-1+2\,p)^{-2}\,\frac{1}{\sqrt{-\frac{np(-1+p)}{(-1+2\,p)^3}}} \quad .$$

The *kurtosis* (aka scaled fourth moment about the mean) is

$$\alpha_4[X_{n,p}] \,=\, \frac{-4\,p^4+(6\,n+8)\,p^3+(-9\,n+6)\,p^2+(3\,n-10)\,p-1}{np\,(-1+p)\,(-1+2\,p)} \quad .$$

For the 5-th through 10-th scaled moments, see the output file

`http://sites.math.rutgers.edu/~zeilberg/tokhniot/oGenPileGames1.txt` .

**The Two Player version for the $(1,-1)$ case**

Using Lagrange inversion (see [Z4] for a lucid statement and proof) or otherwise, it is easy to see that the probability of reaching $m$ dollars for the first time after exactly $n$ rounds, in a solitaire game where the probability of winning a dollar is $p$ and the probability of losing a dollar is $1-p$, let's call it $b_{n,m}$ is

$$b_{n,m} = \frac{m\,(2\,n+m-1)!\,p^{n+m}\,(1-p)^n}{n!\,(n+m)!} \quad .$$

Suppose that two players take turns and whoever reaches $m$ dollars first is declared the winner. As before, the probability of winning the game for the player whose turn is to move is $a(m) = (1+f(m))/2$, where

$$f(m) = \sum_{n=1}^{\infty} b_{n,m}^2 \quad .$$

Using the Zeilberger algorithm once again we have the next computer-generated proposition.

**Proposition 7**: In the two player version game with a **fair** coin, i.e. the probability of winning a dollar and losing a dollar are both $\frac{1}{2}$, the winning probability of the player whose turn is to move is $(1+f(m))/2$ where $f(m)$ satisfies the second-order recurrence

$$\left(2\,m^2+5\,m+2\right) f\,(m+2) + \left(-12\,m^2-24\,m-10\right) f\,(m+1) + \left(2\,m^2+3\,m\right) f\,(m) = -\frac{8}{\pi} \quad ,$$

subject to the initial conditions

$$f\,(1) = -\frac{-4+\pi}{\pi} \quad , \quad f\,(2) = -\frac{-16+5\,\pi}{\pi} \quad .$$

For the loaded case, where $p > \frac{1}{2}$, we have the next proposition.

**Proposition 8**: In the two player version game with the probability of winning a dollar is $p$ and losing a dollar is $1-p$ , provided $\frac{1}{2} < p < 1$, the winning probability of the player whose turn is to move is $(1+f(m))/2$ where $f(m)$ satisfies the fourth-order recurrence

$$m\,(-1+p)^4\,(m-3)\,f\,(m) - (-1+p)^2\left(2\,m^2-7\,m+4\right) f\,(m-1)$$
$$+\left(-2\,m^2p^4+4\,m^2p^3+8\,mp^4-2\,m^2p^2-16\,mp^3-4\,p^4+8\,mp^2+8\,p^3+m^2-4\,p^2-4\,m+4\right) f\,(m-2)$$
$$-p^2\left(2\,m^2-9\,m+8\right) f\,(m-3) + p^4\,(m-1)\,(m-4)\,f\,(m-4) = 0 \quad ,$$

subject to the appropriate initial conditions.

**Winning a dollar or losing k dollars**

Now let's generalize to the gambling game where, as before, you start with a capital of 0 dollars, but now at each round you **win** a dollar with probability $p$ or **lose** $k$ dollars with probability $1-p$, and the game ends as soon as you owe 1 dollar. Very soon we will treat the more general case where the goal is to exit with $m$ dollars, but for now let's consider the case of $m=1$.

In order to guarantee that the game ends, the expected gain of a single round, $p\cdot 1-(1-p)\cdot k = (k+1)\,p - k$ should be positive. So we will assume that $p > \frac{k}{k+1}$. In the border-line case $p = \frac{k}{k+1}$ the game still ends with probability 1, but its expected duration is infinite.

Now the **alphabet** is $\{1, -k\}$, and we will try to adapt the above argument that workded for the classical case. Let's abbreviate $\bar{k} := -k$. Now the steps are $(1, 1)$ and $(1, -k)$.

Let's study the *anatomy* of such words (histories) or, equivalently, paths. Obviously all these words are of length $n(k+1)+1$, for some non-negative integer $n$, and they all end with 1. So we can write, for *any* history $W$,

$$W = U\,1 \quad ,$$

where $U$ is a word that sums to 0, all whose partial sums are non-positive. we will call such words $(1, -k)$-*Dyck words*.

Let's analyze such a $(1, -k)$-Dyck word $U$ or rather its corresponding path from $(0, 0)$ to $((k+1)n, 0)$, say. Of course, it may be the empty word, but if it is not, let $(r(k+1), 0)$ $0 < r \leq n$ be the **first** time that it hits the $x$-axis. Then we can write

$$U = U_1\,U_2 \quad ,$$

where $U_2$ is another arbitrary $(1, -k)$-Dyck word, but $U_1$ has the special property that all its partial sums (except the 0-th and the last) are **strictly negative**, or in terms of its path, except for its starting and ending points, they lie **strictly** below the $x$-axis. Such a word must **necessarily** start with a $\bar{k}$ and end with a 1, but to recover the 'debt' of $k$, must regain these lost $k$ dollars, one dollar at a time, so it may be written as $\bar{k}\,(U_3 1)^k$, where $U_3$ is an arbitrary $(1, -k)$-Dyck word. Conversely, for any such word $U_3$, $\bar{k}\ (U_3\,1)^k$ is such a strictly below the $x$-axis word. So we have the (context-free) *grammar*

$$U = EmptyWord \,\vee\, \bar{k}\ (U\,1)^k U \quad , \qquad ((1, -k) - DyckGrammar)$$

where now $U$ stands for 'an arbitrary $(1, -k)$-Dyck word'.

Let $F(x_{-k}, x_1)$ be the *weight-enumerator* for all such words. Applying the *weight* operation, we get that $F = F(x_{-k}, x_1)$ satisfies

$$F = 1 \,+\, (x_{-k} x_1^k)\, F^{k+1} \quad .$$

Abbreviating $X := x_{-k}\, x_1^k$, this can be written

$$F = 1 \,+\, X\, F^{k+1} \quad .$$

When $k = 2$ and $k = 3$, we can solve these equations 'explicitly' using 'radicals', thanks to Cardano and Ferrari, but thanks to Abel, Ruffini, and Galois we know that we can **not** do it for $k \geq 4$. Even the 'explicit' solutions for $k = 2$ and $k = 3$ are not very useful. On the other hand, thanks to *Lagrange inversion* (see,e.g. [Z4]) we can find the Maclaurin expansion explicitly.

$$F(X) \,=\, \sum_{m=0}^{\infty} \frac{((k+1)\,m)!}{m!\,(km\,+\,1)!} X^m \quad ,$$

featuring the **Fuss-Catalan** numbers $C_{k,m} = \frac{((k+1)\,m)!}{m!\,(km+1)!}$ .

It follows that the weight-enumerator of words in $\{-k, 1\}$ that add-up to 1, and such that the proper-partial sums are all non-positive is $F(x_{-k}x_1^k)\,x_1$, since the last letter must be 1.

Equivalently (and that's is our actual object of interest) the number of words with $m$ '$-k$' and $mk+1$ '1' whose proper-partial sums are all non-positive equals the Fuss-Catalan number $C_{k,m}$. This can be also proved by adapting the [DM] proof. There are $\binom{mk+1+m}{m}$ words altogether, and for each of these its $mk+1+m$ cyclic shifts are all different, and exactly one of them is a 'good' word, hence there are $\frac{1}{mk+1+m}\binom{mk+1+m}{m} = C_{k,m}$ such words.

Since, in order to exit with $n$ dollars , we must gain one dollar, $n$ times, the weight-enumerator of words that reach $n$ for the first time is $(F(x_{-k}x_1^k)\,x_1)^n$.

So far we did enumerative combinatorics. To convert it to probability, we plug-in the above $x_1 = p\,t$ and $x_{-k} = (1-p)\,t$. Using implicit differentiation, we can compute the expectation, variance, and higher moments. Since in this case we do not encounter $0/0$, all the moments are **rational** functions of $p$. In particular, if the number $p$ is rational, all the quantities are rational numbers.

Using implicit differentiation, for **symbolic** $k$ and **symbolic** $p$ and **symbolic** $n$, our beloved computer generated the next proposition.

**Proposition 9**: Suppose that at each round, you win a dollar with probability $p$ and lose $k$ dollars with probability $1-p$, and you quit as soon as you reach $n$ dollars. If $p > k/(k+1)$, then, of course, sooner or later you will reach your goal. How long should it take? Denote by $X_{n,k,p}$ the random variable, 'number of moves until reaching $n$ dollars'. We have the following facts.

Let $g(t)$ be the formal power series, in $t$, satisfying the algebraic equation

$$g\,(t) - 1 - p^k\,(1-p)\,t^{k+1}\,g(t)^{k+1} \;=\; 0 \quad .$$

The probability generating function of $X_{n,k,p}$ is

$$(\,p\,t\,g(t)\,)^n \quad .$$

By implicit differentiation, followed by substituting $t = 1$, we can compute any desired derivative, and hence the expectation, variance, and higher moments. We have

$$E[X_{n,k,p}] \;=\; \frac{n}{(p-1)k+p} \quad ,$$

[as *expected* (npi), since the expected gain in one move is $(p-1)k+p$ ]. The variance is given by

$$Var[X_{n,k,p}] \;=\; -\frac{np\,(k+1)^2\,(p-1)}{((p-1)\,k+p)^3} \quad .$$

The *skewness* (aka 'third scaled-moment about the mean') is

$$\alpha_3[X_{n,k,p}] \;=\; -\,(k+1)\left(kp^2+p^2-k-2\,p\right)(kp-k+p)^{-2}\,\frac{1}{\sqrt{-\frac{np(k+1)^2(p-1)}{((p-1)k+p)^3}}} \quad .$$

The *kurtosis* (aka 'fourth scaled-moment about the mean') is

$$\alpha_3[X_{n,k,p}] =$$

$$\frac{-(k+1)^2 p^4 - 2\,(k+1)\left(k-\frac{3}{2}n-3\right)p^3 + \left(6\,k^2+(-6\,n+6)\,k-3\,n-6\right)p^2 - 2\,k\left(k-\frac{3}{2}n+4\right)p - k^2}{np\,(p-1)\,(p\,(k+1)-k)} \quad .$$

For the scaled fifth and sixth moments, see the output file

`http://sites.math.rutgers.edu/ zeilberg/tokhniot/oGenPileGames2.txt` .

**The Two Player version for the** $(1,-k)$ **case**

Since the probability mess function is explicit, given in terms of the Fuss-Catalan numbers, we can use the Zeilberger algorithm to compute recurrences for the probability of the first player winning, for **symbolic** $n$, and **symbolic** $p$ (assuming that it is larger than $\frac{k}{k+1}$). Alas, we can **not** do it for symbolic $k$, since the Fuss-Catalan numbers are not bi-holonomic in both $n$ and $k$.

For the case $k=2$ we have the next proposition.

**Proposition 10**: In the two player version game, if the probability of winning a dollar is $p$ and of losing two dollars is $1-p$ , provided $\frac{2}{3} < p < 1$, the probability of the player whose turn is to move of winning the game is $(1+f(m))/2$ where $f(m)$ satisfies the sixth-order linear recurrence

$$m\,(p-1)^4\,(m-5)\,f(m)$$

$$-2\,(p-1)^2\left(m^2-6\,m+6\right)f(m-2) - p^2\,(p-1)^2\left(2\,m^2-13\,m+12\right)f(m-3)$$

$$+(m-3)\,(m-4)\,f(m-4) - p^2\left(2\,m^2-15\,m+24\right)f(m-5) + p^4\,(m-2)\,(m-6)\,f(m-6) = 0 \quad ,$$

subject to the appropriate initial conditions.

For the case $k=3$ we have the next proposition.

**Proposition 11**: In the two player version game, if the probability of winning a dollar is $p$ and of losing three dollars is $1-p$ , provided $\frac{3}{4} < p < 1$, the probability of the player whose turn is to move of winning the game is $(1+f(m))/2$ where $f(m)$ satisfies the eighth-order linear recurrence

$$m\,(p-1)^4\,(m-7)\,f(m) - (p-1)^2\left(2\,m^2-17\,m+24\right)f(m-3)$$

$$-2\,p^2\,(p-1)^2\left(m^2-9\,m+12\right)f(m-4) + (m-4)\,(m-6)\,f(m-6)$$

$$-p^2\left(2\,m^2-21\,m+48\right)f(m-7) + p^4\,(m-3)\,(m-8)\,f(m-8) = 0 \quad ,$$

subject to the appropriate initial conditions.

For the case $k=4$ we have the next proposition.

**Proposition 12**: In the two player version game, if the probability of winning a dollar is $p$ and of losing four dollars is $1-p$ , provided $\frac{4}{5} < p < 1$, the probability of the player whose turn is to move of winning the game is $(1+f(m))/2$ where $f(m)$ satisfies the tenth-order linear recurrence

$$m\,(p-1)^4\,(m-9)\,f(m) - 2\,(p-1)^2\left(m^2-11\,m+20\right)f(m-4)$$

$$-p^2\,(p-1)^2\left(2\,m^2-23\,m+40\right)f(m-5)+(m-5)\,(m-8)\,f(m-8)$$

$$-p^2\left(2\,m^2-27\,m+80\right)f(m-9)+p^4\,(m-4)\,(m-10)\,f(m-10)=0 \quad ,$$

subject to the appropriate initial conditions.

For the case $k=5$ we have the next proposition.

**Proposition 13**: In the two player version game, if the probability of winning a dollar is $p$ and of losing five dollars is $1-p$ , provided $\frac{5}{6} < p < 1$, the probability of the player whose turn is to move of winning the game is $(1+f(m))/2$ where $f(m)$ satisfies the $12^{th}$-order linear recurrence

$$m\,(p-1)^4\,(m-11)\,f(m) - (p-1)^2\left(2\,m^2-27\,m+60\right)f(m-5)$$

$$-2\,p^2\,(p-1)^2\left(m^2-14\,m+30\right)f(m-6)+(m-6)\,(m-10)\,f(m-10)$$

$$-p^2\left(2\,m^2-33\,m+120\right)f(m-11)+p^4\,(m-5)\,(m-12)\,f(m-12)=0 \quad ,$$

subject to the appropriate initial conditions.

**Winning k dollars or losing one dollar**

This case is more complicated than the previous one, and we will have to treat one $k$ at a time even for the expectation. Also, we only consider the case of reaching at least one dollar for the first time, rather than the more general case of reaching $n$ dollars for the first time.

Now our **alphabet** is $\{\,k\,,\,-1\,\}$ and, in terms of lattice paths, the atomic steps are $(1,k)$ and $(1,-1)$.

Since the last step of such a path must be $(1,k)$ it can terminate at $y=k$, or $y=k-1, \ldots, y=1$, so we are forced to consider, in addition to $U_{0,0}$ the set of paths that start at $y=0$ and end at $y=0$ and never go above the $x$-axis, also $U_{0,-1}$ the set of paths that start at $y=0$ and end at $y=-1$ and never go above the $x$-axis, all the way to $U_{0,-(k-1)}$, the set of paths that start at $y=0$ and end at $y=-(k-1)$ and never go above the $x$-axis.

Such a word looks like

$$U_{0,0}\,k \vee U_{0,-1}\,k \vee \quad \ldots \quad \vee U_{0,-(k-1)}\,k \quad .$$

Let $U := U_{0,0}$. Then the weight-enumerator of $U$ is $F(x_k x_{-1}^k)$ where $F(X)$ is as above, the solution of

$$F(X) \,=\, 1 \,+\, XF(X)^{k+1} \quad .$$

It can be seen that $U_{0,-r} = (\bar{1}\, U_{0,0})^r$, hence its weight-enumerator is $(x_{-1}\, F(X))^r$.

Substituting for $x_{-1} = p\,t$ and $x_k = (1-p)\,t$, we get the following human-generated proposition.

**Proposition 14**: Suppose that at each round, you lose one dollar with probability $p$ and win $k$ dollars with probability $1-p$, and you quit as soon as you reach at least 1 dollar. If $0 < p < \frac{k}{k+1}$ then, of course, sooner or later, you will reach your goal. Let $g(t)$, be the formal power series, in $t$, satisfying the algebraic equation

$$g(t) \,-\, 1 \,-\, p^k\,(1-p)\,t^{k+1}\,g(t)^{k+1} \,=\, 0 \quad .$$

The probability generating function, let's call it $f(t)$, for the number of rounds until having a positive capital is

$$f(t) \,=\, (1-p)\,t\,g(t)\,\sum_{i=0}^{k-1} (p\,t\,g(t))^i \quad .$$

If you will apply implicit differentiation to the defining equation of $g(t)$, and then express $f'(t)$ in terms of $g(t)$ and $g'(t)$ and then plug-in $t=1$, you will get $0/0$. It turns out that the expressions for the expectation, variance, and higher moments are no longer rational functions of $p$, but are roots of **algebraic** equations. The reason is that when $t=1$, 1 is a double (or higher-order) root of the defining equation for the probability itself $f(1)=1$.

Since Maple knows how to differenitate, both explicitly and implicitly, our beloved computer can handle it all automatically, and get explicit algebraic equation for symbolic $p$, or specific algebraic numbers for specific $p < \frac{k}{k+1}$, alas only for **one $k$ at a time**.

We have the following computer-generated proposition for the case $k=2$, i.e. for the gambling options $\{-1, 2\}$, with $Pr(-1) = p$ and $Pr(2) = 1-p$.

**Proposition 15**: Let $X$ be the random variable 'number of rounds until you reach positive capital' if you start at 0, and at each round, you lose 1 dollar with probability $p$ and win 2 dollars with probability $1-p$. Assume that $p < \frac{2}{3}$.

The expectation is given by

$$E[X] \,=\, \frac{3\,p + \sqrt{-\,(3\,p+1)\,(-1+p)} - 1}{2\,p\,(2\,-\,3\,p)}$$

For the variance, and third through the sixth moment, see

`http://sites.math.rutgers.edu/z̃eilberg/tokhniot/oGenPileGames3.txt` .

Note that for the most interesting case, $p = \frac{1}{2}$, the expectation is the **beautiful number** $1+\sqrt{5}$ (twice the golden ratio). This is so nice that we will single it out.

**Beautiful Corollary**: If a one-dimensional random walker starts at 0 and moves **one step back** with probability $\frac{1}{2}$ and **two steps forward** with probability $\frac{1}{2}$ and keeps going until he is at a location $\geq 1$ for the first time, the expected number of steps that he takes is twice the Golden Ratio, i.e. $1+\sqrt{5}$.

For $k \geq 3$ and *symbolic* $p$, things get too complicated to reproduce here, so let's just mention the expectations for a few cases for the most interesting case, $p=\frac{1}{2}$.

$k=3$: The expected duration of a random walk with $Pr(-1)=Pr(3)=\frac{1}{2}$ until reaching a location $\geq 1$ for the first time is the positive root of

$$x^3-4\,x-4=0 \quad ,$$

that equals $2.382975767906237494\ldots$ .

$k=4$: The expected duration of a random walk with $Pr(-1)=Pr(4)=\frac{1}{2}$ until reaching a location $\geq 1$ for the first time is the positive root of

$$3\,x^4+4\,x^3-8\,x^2-24\,x-16 \;=\; 0 \quad ,$$

that equals $2.1561901553356811691\ldots$ .

$k=5$: The expected duration of a random walk with $Pr(-1)=Pr(5)=\frac{1}{2}$ until reaching a location $\geq 1$ for the first time is the positive root of

$$2\,x^5+5\,x^4-20\,x^2-32\,x-16 \;=\; 0 \quad ,$$

that equals $2.07050432323944926\ldots$ .

$k=6$: The expected duration of a random walk with $Pr(-1)=Pr(6)=\frac{1}{2}$ until reaching a location $\geq 1$ for the first time is the positive root of

$$5\,x^6+18\,x^5+20\,x^4-40\,x^3-144\,x^2-160\,x-64 \;=\; 0 \quad ,$$

that equals $2.0333823565252879532\ldots$ .

$k=7$: The expected duration of a random walk with $Pr(-1)=Pr(7)=\frac{1}{2}$ until reaching a location $\geq 1$ for the first time is the positive root of

$$3\,x^7+14\,x^6+28\,x^5-112\,x^3-224\,x^2-192\,x-64 \;=\; 0 \quad ,$$

that equals $2.0162018012796575781\ldots$ .

$k=8$: The expected duration of a random walk with $Pr(-1)=Pr(8)=\frac{1}{2}$ until reaching a location $\geq 1$ for the first time is the positive root of

$$7\,x^8+40\,x^7+112\,x^6+112\,x^5-224\,x^4-896\,x^3-1280\,x^2-896\,x-256 \;=\; 0 \quad ,$$

that equals $2.00796926912597191\ldots$ .

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