Yet Another Proof for the Enumeration of Labelled Trees

Based on a comment of Herb Wilf (spelled out by D. Zeilberger)

[Exclusive for DZ's mailing list, and his ftp and www forum.]

In [1], a very short and elementary proof of Abel's identity was given, using the methods introduced in [2]. For the sake of completeness we reproduce the statement and proof.

Theorem: For $n \geq 0$:

$$\sum_{k=0}^{n} \binom{n}{k} (r+k)^{k-1}(s-k)^{n-k} = \frac{(r+s)^n}{r} \qquad (1)$$

Proof ([1]): Let $F_{n,k}(r,s)$ and $a_n(r,s)$ denote, respectively, the summand and sum on the LHS of (1), and let $G_{n,k} := (s-n)\binom{n-1}{k-1}(r+k)^{k-1}(s-k)^{n-k-1}$. Since

$$F_{n,k}(r,s) - sF_{n-1,k}(r,s) - (n+r)F_{n-1,k}(r+1,s-1) + (n-1)(r+s)F_{n-2,k}(r+1,s-1) = G_{n,k} - G_{n,k+1}\,,$$

(check!), we have by summing from $k=0$ to $k=n$, thanks to the telescoping on the right:

$$a_n(r,s) - sa_{n-1}(r,s) - (n+r)a_{n-1}(r+1,s-1) + (n-1)(r+s)a_{n-2}(r+1,s-1) = 0.$$

Since $(r+s)^n \cdot r^{-1}$ also satisfies this recurrence (check!) with the same initial conditions $a_0(r,s) = r^{-1}$ and $a_1(r,s) = (r+s) \cdot r^{-1}$, (1) follows.□

Now, letting $n \to n-2$, $r := 1$, and $s := n-1$, and setting $b_n := n^{n-2}$, one obtains the recurrence:

$$b_n = \sum_{k=0}^{n-2} \binom{n-2}{k} b_{k+1}[(n-k-1)b_{n-k-1}]. \qquad (2)$$

Let t_n be the number of labelled trees on n vertices, then:

$$t_n = \sum_{k=0}^{n-2} \binom{n-2}{k} t_{k+1}[(n-k-1)t_{n-k-1}]. \qquad (3)$$

Indeed every labelled tree T on $\{1,2,\ldots,n\}$ gives rise to a unique triple (T', T'', S), where T'' is the rooted tree to which the vertex 2 belongs, in the forest resulting from deleting 1 (rooted at the vertex connected to 1), T' is the tree obtained from T by deleting T'', and S is the set of labels (in addition to 1) participating in T'. Now sum over all possible $k := |S|$, to get (3).

Since $b_1 = t_1$, and b_n and t_n satisfy the same recurrence, it follows that we have the (n^{n-2})th proof of Cayley's theorem.

References

1. S. B. Ekhad and J. Majewicz *A short WZ-style proof of Abel's identity*, preprint. (available via anonymous ftp to ftp.math.temple.edu in directory pub/jmaj)

2. J. Majewicz *WZ-type certification procedures and Sister Celine's technique for Abel-type sums*, preprint (available via anonymous ftp to ftp.math.temple.edu in directory pub/jmaj).