Yet Another Proof for the Enumeration of Labelled Trees

Based on a comment of Herb Wilf (spelled out by D. Zeilberger)

[Exclusive for DZ’s mailing list, and his ftp and www forum.]

In [1], a very short and elementary proof of Abel’s identity was given, using the methods introduced in [2]. For the sake of completeness we reproduce the statement and proof.

**Theorem:** For \( n \geq 0 \):
\[
\sum_{k=0}^{n} \binom{n}{k} (r + k)^{k-1}(s - k)^{n-k} = \frac{(r + s)^n}{r}
\]  \hspace{1cm} (1)

**Proof** ([1]): Let \( F_{n,k}(r,s) \) and \( a_n(r,s) \) denote, respectively, the summand and sum on the LHS of (1), and let \( G_{n,k} := (s - n) \binom{n-1}{k}(r + k)^{k-1}(s - k)^{n-k-1} \). Since
\[
F_{n,k}(r,s) - sF_{n-1,k}(r,s) - (n+r)F_{n-1,k}(r+1,s-1) + (n-1)(r+s)F_{n-2,k}(r+1,s-1) = G_{n,k} - G_{n,k+1},
\]
(check!), we have by summing from \( k = 0 \) to \( k = n \), thanks to the telescoping on the right:
\[
a_n(r,s) - sa_{n-1}(r+1,s-1) + (n-1)(r+s)a_{n-2}(r+1,s-1) = 0.
\]
Since \((r+s)^n \cdot r^{-1}\) also satisfies this recurrence (check!) with the same initial conditions \( a_0(r,s) = r^{-1} \) and \( a_1(r,s) = (r+s) \cdot r^{-1} \), (1) follows \( \square \).

Now, letting \( n \rightarrow n-2 \), \( r := 1 \), and \( s := n-1 \), and setting \( b_n := n^{n-2} \), one obtains the recurrence:
\[
b_n = \sum_{k=0}^{n-2} \binom{n-2}{k} b_{k+1}[(n-k-1)b_{n-k-1}].
\]  \hspace{1cm} (2)

Let \( t_n \) be the number of labelled trees on \( n \) vertices, then:
\[
t_n = \sum_{k=0}^{n-2} \binom{n-2}{k} t_{k+1}[(n-k-1)t_{n-k-1}].
\]  \hspace{1cm} (3)

Indeed every labelled tree \( T \) on \( \{1,2,\ldots,n\} \) gives rise to a unique triple \((T',T'',S)\), where \( T'' \) is the rooted tree to which the vertex 2 belongs, in the forest resulting from deleting 1 (rooted at the vertex connected to 1), \( T' \) is the tree obtained from \( T \) by deleting \( T'' \), and \( S \) is the set of labels (in addition to 1) participating in \( T' \). Now sum over all possible \( k := |S| \), to get (3).

Since \( b_1 = t_1 \), and \( b_n \) and \( t_n \) satisfy the same recurrence, it follows that we have the \((n^{n-2})\)th proof of Cayley’s theorem.

**References**


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