COMBINATORIAL PROOFS OF CAPELLI'S AND TURNBULL'S IDENTITIES FROM CLASSICAL INVARIANT THEORY

BY

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0. Introduction. Capelli's [C] identity plays a prominent role in Weyl's [W] approach to Classical Invariant Theory. Capelli's identity was recently considered by Howe [H] and Howe and Umeda [H-U]. Howe [H] gave an insightful representation-theoretic proof of Capelli's identity, and a similar approach was used in [H-U] to prove Turnbull's [T] symmetric analog, as well as a new anti-symmetric analog, that was discovered independently by Kostant and Sahi [K-S]. The Capelli, Turnbulll, and Howe-Umeda-Kostant-Sahi identities immediately imply, and were inspired by, identities of Cayley (see [T1]), Garding [G], and Shimura [S], respectively.

In this paper, we give short combinatorial proofs of Capelli's and Turnbull's identities, and raise the hope that someone else will use our approach to prove the new Howe-Umeda-Kostant-Sahi identity.

1. The Capelli Identity. Throughout this paper $x_{i,j}$ are mutually commuting indeterminates ("positions"), as are $o_{i,j}$ ("momenta"), and they interact with each other via the "uncertainty principle"

$$p_{ij}x_{ij} - x_{ij}p_{ij} = h,$$

and otherwise $x_{i,j}$ commutes with all the $p_{k,l}$ if $(i,j) \neq (k,l)$. Of course, one can take $p_{i,j} := h(\partial/\partial x_{i,j})$. Set $X = (x_{ij})$, $P = (p_{ij})$ $(1 \le i,j \le n)$.

Capelli's Identity. *For each positive integer n and for $1 \le i,j \le n$ let*

$$A_{ij} = \sum_{k=1}^{n} x_{ki}p_{kj} + h(n-i)\delta_{ij}. \tag{1.1}$$

Then

$$\sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma)\, A_{\sigma 1,1}\ldots A_{\sigma n,n} = \det X.\det P. \tag{CAP}$$

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Remark 1. The Capelli identity can be viewed as a "quantum analog" ot the classical Cauchy-Binet identity $\det XP = \det X . \det P$, when the entries of X and P commute, and indeed reduces to it when $h = 0$. The matrix A is $X^t P$, with "quantum correction" $h(n-i)\delta_{i,j}$.

Remark 2. Note that since not all indeterminates commute, it is necessary to define order in the definition of the determinant of A. It turns out that the determinant is to be evaluated by "column expansion" rather than "row expansion," which is reflected in the left side of (CAP).

Combinatorial Proof of Capelli's Identity. We will first figure out, step by step, what the combinatorial objects that are being weight-enumerated by the left side of (CAP). Then we will decide who are the "bad guys" and will find an involution that preserves the absolute value of the weight, but reverses the sign. The weight-enumerator of the good guys will turn out to be counted by the right side of (CAP).

First we have to represent each A_{ij} as a generating polynomial over a particular set of combinatorial objects: consider the 4-tuples (i, j, k, l) where i, j, $k = 1, 2, \ldots, n$ and $l = 0, 1$. For $i \neq j$ define $\mathcal{A}_{ij}$ as the set of all 4-tuples (a, b, c, d) such that $a = i$, $c = j$, $d = 0$ and $b = 1, 2, \ldots, n$. Next define $\mathcal{A}_{ii}$ as the set of all 4-tuples (a, b, c, d) such that $a = c = i$, and either $d = 0$ and $b = 1, 2, \ldots, n$, or $d = 1$ and $b = i+1, \ldots, n$. Finally, let

$$w(a, b, c, d) = \begin{cases} x_{ba}\, p_{bc}, & \text{if } d = 0; \\ h, & \text{if } d = 1 \text{ (and } a = c). \end{cases}$$

We can then rewrite: $A_{ij} = \sum w(a, b, c, d)$, where (a, b, c, d) runs over all $\mathcal{A}_{ij}$. Hence

$$\text{(1.2)} \quad \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma)\, A_{\sigma 1,1} \ldots A_{\sigma n,n} \\ = \sum \operatorname{sgn}(a)\, w(a_1, b_1, c_1, d_1) \ldots w(a_n, b_n, c_n, d_n),$$

where the sum is over all sequences $(a_1, b_1, c_1, d_1, \ldots, a_n, b_n, c_n, d_n)$ satisfying the properties:

1) $a = (a_1, \ldots, a_n)$ is a permutation;
2) $(c_1, \ldots, c_n) = (1, \ldots, n)$;
3) $d_i = 0$ or 1 $(i = 1, \ldots, n)$;
4) the b_i's are arbitrary $(1 \leq b_i \leq n)$ with the sole condition that when $d_i = 1$, then $a_i = i = c_i$ and $i + 1 \leq b_i \leq n$.

It then suffices to consider the set $\mathcal{A}$ of all $4 \times n$-matrices

$$G = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \\ c_1 & c_2 & \ldots & c_n \\ d_1 & d_2 & \ldots & d_n \end{pmatrix}$$

that satisfy the forementioned 1) to 4) properties and define the *weight* of G as

$$w(G) = \operatorname{sgn}(a) \prod_{i=1}^{n} (x_{b_i,a_i}\, p_{b_i,i}(1-d_i) + h d_i). \tag{1.3}$$

Then the (1.2) sum may be expressed as:

$$\sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma)\, A_{\sigma 1,1} \ldots A_{\sigma n,n} = \sum_{G \in \mathcal{A}} w(G).$$

If there is no pair (i,j) such that $1 \le i < j \le n$, $d_i = d_j = 0$ and $(b_i, i) = (b_j, a_j)$, say that G is *not linkable*. Its weight can be expressed as a monomial $\operatorname{sgn}(a)\, h^\alpha \prod x^\beta \prod p^\gamma$, where all the x's are written before all the p's, by using the commutation rule. If there exists such a pair $(b_i, i) = (b_j, a_j)$, the matrix G is said to be *linkable* . The product $x_{b_i,a_i} p_{b_i,i} x_{b_j,a_j} p_{b_j,j}$ gives rise to the sum $x_{b_i,a_i} x_{b_i,i} p_{b_i,i} p_{b_j,j} + x_{b_i,a_i} p_{b_j,j} h = x_{b_i,a_i} x_{b_i,i} p_{b_i,i} p_{b_i,j} + x_{b_i,a_i} p_{b_i,j} h$. In the first monomial the commutation $x_{b_i,i}\, p_{b_i,i}$ has been made; in the second monomial the latter product has vanished and been replaced by h. Such a pair (i,j) will be called a *link*, of *source* i and *end* j.

If a linkable matrix has m links $(i_1 < j_1)$, ..., $(i_m < j_m)$, its weight will produce 2^m monomials when the commutation rules are applied to it. Each of those 2^m monomials corresponds to a subset $K = \{k_1, \ldots, k_r\}$ of the set $I = \{i_1, \ldots, i_m\}$ of the link sources. We then have to consider the set of all the pairs (G, K), subject to the previous conditions and define the weights of those pairs as single monomials in such a way that the sum $\sum_K w(G,K)$ will be the weight of G, once all the commutations $px = xp + h$ have been made.

The weight $w(G,K)$ will be defined in the following way: consider the single monomial introduced in (1.3); if i belongs to K, drop $x_{b,i}$ and replace $p_{b_i,i}$ by h; if i belongs to $I \setminus K$, drop $x_{b,i}$ and replace $p_{b_i,i}$ by $x_{b_i,i}\, p_{b_i,i}$. Leave the other terms alike. In other words define the operators:

$$D_i = h \frac{\partial}{\partial p_{b_i,i}} \frac{\partial}{\partial x_{b_i,i}} \qquad \text{and} \qquad \Delta_i = x_{b_i,i} p_{b_i,i} \frac{\partial}{\partial p_{b_i,i}} \frac{\partial}{\partial x_{b_i,i}}.$$

Then let

$$w(G,K) = \Big(\prod_{i \in K} D_i \prod_{i \in I \setminus K} \Delta_i \Big) w(K)$$

For instance, the matrix

$$G = \begin{pmatrix} 4 & 5 & 1 & 8 & 7 & 6 & 9 & 2 & 3 \\ 2 & 8 & 2 & 1 & 8 & 8 & 8 & 8 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

has three links $(1,3)$, $(2,8)$ and $(3,9)$. Its weight, according to (1.3) reads:

$$-x_{2,4}\,p_{2,1}\;x_{8,5}\,p_{8,2}\;x_{2,1}\,p_{2,3}\;x_{1,8}\,p_{1,4}\;x_{8,7}\,p_{8,5}\;h\;x_{8,9}\,p_{8,7}\;x_{8,2}\,p_{8,8}\;x_{2,3}\,p_{3,9}.$$

Now consider the subset $K=\{2\}$ of its link source set $I=\{1,2,3\}$. The weight of $w(G,K)$ is then:

$$-x_{2,4}\,x_{2,1}\,p_{2,1}\;x_{8,5}\;h\;x_{2,3}\,p_{2,3}\;x_{1,8}\,p_{1,4}\;x_{8,7}\,p_{8,5}\;h\;x_{8,9}\,p_{8,7}\;p_{8,8}\;p_{3,9}.$$

The simple drop-add rule just defined guarantees that *no $p_{i,j}$ remains to the left of x_{ij}* in any of the weight $w(G,K)$. After using all the commutations $px=xp+h$ we then get

$$\sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}\sigma\,A_{\sigma1,1}\ldots A_{\sigma n,n}=\sum w(G,K), \tag{1.4}$$

where G runs over all $\mathcal{A}$ and K over all the subsets of the link source set of G.

It is obvious who the *good guys* are: those pairs (G,K) such that G has no 1's on the last row and such that K is empty. The good guys correspond exactly to the members of $\det X^t\,P$, in the classical case, where all the $x_{i,j}$ commute with all the $p_{i,j}$, and obviously their sum is $\det X.\det P$. [A combinatorial proof of which can be found in [Z].] It remains to kill the bad guys, i.e., show that the sum of their weights is zero.

If (G,K) is a bad guy, h occurs in $w(G,K)$ and either there are 1's on the last row of G, or K is non empty. Let $i=i(G,K)$ be the greatest integer $(1\le i\le n-1)$ such that either i a link source belonging to K, or the i-th column has an entry equal to 1 on the last row.

In the first case, let (i,j) be the link of source i; then replace the i-th and the j-th columns as shown in the next display, the other columns remaining intact:

$$G=\begin{pmatrix} a_i & \ldots & i\\ b_i & \ldots & b_i\\ i & \ldots & j\\ 0 & \ldots & 0\end{pmatrix}\mapsto\begin{pmatrix} i & \ldots & a_i\\ j & \ldots & b_i\\ i & \ldots & j\\ 1 & \ldots & 0\end{pmatrix}=G'.$$

The link (i,j) (with $i\in K$) has been suppressed. Let (k,l) be another link of G such that $k\in K$. Then $k<i$ by definition of i. On the other hand, $l\ne j$. If $l\ne i$, then (k,l) remains a link of G'. If $l=i$, then $(b_k,k)=(b_i,a_i)$ and the link (k,i) in G has been replaced by the link (k,j) in G', so that k is still a link source in G'. Accordingly, $K\setminus\{i\}$ is a subset of the link set of G' and it makes sense to define $K'=K\setminus\{i\}$. Also notice that

$$i(G',K')=i(G,K). \tag{1.5}$$

As G and G' differ only by their i-th and j-th columns, the weights of G and G' will have *opposite sign*; furthermore, they will differ only by their i-th and j-th factors, as indicated in the next display:

$$\begin{array}{rcccccc} |w(G)| = & \ldots & x_{b_i,a_i}\, p_{b_i,i} & \ldots & x_{b_i,i}\, p_{b_i,j} & \ldots \\ |w(G')| = & \ldots & h & \ldots & x_{b_i,a_i}\, p_{b_i,j} & \ldots \end{array}$$

[The dots mean that the two words have the same left factor, the same middle factor and the same right factor.] Hence, as i is in K, but not in K', the operator D_i (resp. Δ_i) is to be applied to $w(G)$ (resp. G') in order to get $w(G,K)$ (resp. $w(G,K')$), so that:

$$\begin{array}{rcccccc} |w(G,K)| = & \ldots & x_{b_i,a_i}\, h & \ldots & p_{b_i,j} & \ldots \\ |w(G',K')| = & \ldots & h & \ldots & x_{b_i,a_i}\, p_{b_i,j} & \ldots \end{array}$$

showing that

$$w(G,K) = -w(G',K'). \tag{1.6}$$

In the second case the entries in the i-th column $(i,j,i,1)$ satisfy the inequalities $i+1 \le j \le n$, while the j-th column (on the right of the i-th column) is of the form $(a_j, b_j, j, 0)$. Then define

$$G = \begin{pmatrix} i & \ldots & a_j \\ j & \ldots & b_j \\ i & \ldots & j \\ 1 & \ldots & 0 \end{pmatrix} \mapsto \begin{pmatrix} a_j & \ldots & i \\ b_j & \ldots & b_j \\ i & \ldots & j \\ 0 & \ldots & 0 \end{pmatrix} = G',$$

where only the i-th and j-th columns have been modified. Clearly a new link (i,j) has been created in G'. Let (k,l) be a link of G with $k \in K$. Then $k < i$. If $l = j$, we have $(b_k, k) = (b_j, a_j)$, so that (k,i) is a link of G'. If $l \neq j$, then (k,l) remains a link of G'. Thus $K \cup \{i\}$ is a set of link sources of G'. It then makes sense to define $K' = K \cup \{i\}$. Also notice that relation (1.5) still holds.

As before, $w(G)$ and $w(G)$ have opposite signs. Furthermore

$$\begin{array}{rcccccc} |w(G)| = & \ldots & h & \ldots & x_{b_j,a_j}\, p_{b_j,j} & \ldots \\ |w(G')| = & \ldots & x_{b_j,a_j}\, p_{b_j,i} & \ldots & x_{b_j,i}\, p_{b_j,j} & \ldots \end{array}$$

so that

$$\begin{array}{rcccccc} |w(G,K)| = & \ldots & h & \ldots & x_{b_j,a_j}\, p_{b_j,j} & \ldots \\ |w(G',K')| = & \ldots & x_{b_j,a_j}\, h & \ldots & p_{b_j,j} & \ldots, \end{array}$$

showing that (1.6) also holds.

Taking into account (1.5) it is readily seen that $\omega : (G,K) \mapsto (G',K')$ maps the first case into the second one, and conversely. Applying ω twice gives the original element, so it is an involution. Finally, property (1.6) makes it possible to associate the bad guys into mutually canceling pairs, and hence their total weight is zero. $\square$

2. A Combinatorial Proof of Turnbull's Identity.

Turnbull's Identity. *Let $X = (x_{ij})$, $P = (p_{ij})$ $(1 \le i, j \le n)$ be as before, but now they are symmetric matrices: $x_{i,j} = x_{j,i}$ and $p_{i,j} = p_{j,i}$, their entries satisfying the same commutation rules. Also let $\tilde{P} = (\tilde{p}_{i,j}) := (p_{i,j}(1 + \delta_{i,j}))$. For each positive integer n and for $1 \le i, j \le n$, let*

$$A_{ij} := \sum_{k=1}^{n} x_{ki}\tilde{p}_{kj} + h(n-i)\delta_{ij}. \tag{2.1}$$

Then

$$\sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) A_{\sigma 1,1} \dots A_{\sigma n,n} = \det X . \det \tilde{P}. \tag{TUR}$$

The proof is very similar. However we have to introduce another value for the d_i's to account for the fact that the diagonal terms of $\tilde{P}$ are $2p_{i,i}$. More precisely, for $i \ne j$ we let $\mathcal{T}_{i,j}$ be the set of all 4-tuples (a, b, c, d) such that $a = i$, $c = j$, and either $d = 0$ and $b = 1, 2, \dots, n$, or $d = 2$ and $b = c = j$. In the same way, let $\mathcal{T}_{ii}$ be the set of all 4-tuples (a, b, c, d) such that $a = c = i$, and either $d = 0$ and $b = 1, 2, \dots, n$, or $d = 1$ and $b = i+1, \dots, n$, or $d = 2$ and $b = c = i$. Finally, let

$$w(a, b, c, d) = \begin{cases} x_{ba}\, p_{bc}, & \text{if } d = 0 \text{ or } 2; \\ h, & \text{if } d = 1 \text{ (and } a = c). \end{cases}$$

Next consider the set $\mathcal{T}$ of all $4 \times n$-matrices

$$G = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \\ c_1 & c_2 & \dots & c_n \\ d_1 & d_2 & \dots & d_n \end{pmatrix}$$

satisfying the properties:

1) $a = (a_1, \dots, a_n)$ is a permutation;
2) $(c_1, \dots, c_n) = (1, \dots, n)$;
3) $d_i = 0$, 1 or 2 $(i = 1, \dots, n)$;
4) $b_i = \begin{cases} 1, \dots, \text{ or } n, & \text{when } d_i = 0; \\ i+1, \dots, \text{ or } n, \text{ and } a_i = c_i = i, & \text{when } d_i = 1; \\ c_i = i, & \text{when } d_i = 2. \end{cases}$

Then we have

$$\sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma)\, A_{\sigma 1,1} \dots A_{\sigma n,n} = \sum_{G \in \mathcal{T}} w(G),$$

where the weight $w(G)$ is defined as in (1.3) under the restriction that the d_i's are to be taken mod 2.

Now to take the symmetry of P and X into account the definition of a *link* has to be slightly modified. Say that a pair (i, j) is a *link* in G, if $1 \le i < j \le n$, $d_i \equiv d_j \equiv 0 \pmod 2$ and either $(b_i, i) = (b_j, a_j)$, or $(b_i, i) = (a_j, b_j)$. In the Capelli case the mapping $i \mapsto j$ (with $1 \le i < j \le n$ and $(b_i, i) = (b_j, a_j)$) set up a natural bijection of the source set onto the end set. Furthermore, if the latter sets were of cardinality m, the weight of G gave rise to a polynomial with 2^m terms. It is no longer the case in the Turnbull case. For instance, if a matrix G is of the form

$$G = \begin{pmatrix} \dots & . & \dots & . & \dots & b_i & \dots & i & \dots \\ \dots & b_i & \dots & i & \dots & i & \dots & b_i & \dots \\ \dots & i & \dots & b_i (= k) & \dots & j & \dots & l & \dots \\ \dots & d_i & \dots & d_k & \dots & d_j & \dots & d_l & \dots \end{pmatrix}$$

with $d_i \equiv d_k \equiv d_j \equiv d_l \equiv 0 \pmod 2$, the weight of G will involve the factor

$$p_{b_i, i} p_{i, b_i} x_{i, b_i} x_{b_i, i} = ppxx$$

(by dropping the subscripts). The expansion of the latter monomial will yield

$$ppxx = xxpp + 4hxp + 2h^2.$$

With the term "$xxpp$" all the commutations have been made; say that *no link* remains. One link remains unused to obtain each one of the next four terms "hxp," i.e., (i, j), (i, l), (k, j), (k, l). Finally, the two pairs of links $\{(i, j), (k, l)\}$ and $\{(i, l), (k, j)\}$ remain unused to produce the last term "$2h^2$."

Accordingly, each of the term in the expansion of the weight $w(G)$ (once all the commutations $px = xp + h$ have been made) corresponds to a subset $K = \{(i_1, j_1), \dots, (i_r, j_r)\}$ of the link set of G having the property that all the i_k's (resp. all the j_k's) are *distinct*. Let $w(G, K)$ denote the term corresponding to K in the expansion. We will then have

$$w(G) = \sum_K w(G, K).$$

As before the product $\det X . \det \tilde{P}$ is the sum $\sum w(G, K)$ with K empty and no entry equal to 1 on the last row of G. If (G, K) does not verify the last two conditions, let $i = i(G, K)$ be the greatest integer $(1 \le i \le n - 1)$ such that one of the following conditions holds:

1) $d_i = 0$ and i is the source of a link (i, j) belonging to K such that $(b_i, i) = (b_j, a_j)$;
2) the i-th column has an entry equal to 1 on the last row;
3) $d_i = 0$ or 2 and i is the source of a link (i, j) belonging to K such that $(b_i, i) = (a_j, b_j)$ and case 1 does not hold.

For cases 1 and 2 the involution $\omega : (G, K) \mapsto (G', K')$ is defined as follows:

Case 1:

$$G = \left(\begin{array}{ccc} a_i & \dots & i \\ b_i & \dots & b_i \\ i & \dots & j \\ 0 & \dots & d_j \end{array}\right) \mapsto \left(\begin{array}{ccc} i & \dots & a_i \\ j & \dots & b_i \\ i & \dots & j \\ 1 & \dots & d_j \end{array}\right) = G'.$$

Case 2:

$$G = \left(\begin{array}{ccc} i & \dots & a_j \\ j & \dots & b_j \\ i & \dots & j \\ 1 & \dots & d_j \end{array}\right) \mapsto \left(\begin{array}{ccc} a_j & \dots & i \\ b_j & \dots & b_j \\ i & \dots & j \\ 0 & \dots & d_j \end{array}\right) = G'.$$

Notice that $d_j = 0$ or 2 and when $d_j = 2$, the matrix G' also belongs to $\mathcal{T}$.

In case 1 the link (i, j) has been suppressed. Let (k, l) be a link in G with $(k, l) \in K$. Then $k < i$ and $l \neq j$ because of our definition of K. If $l \neq i$, then (k, l) remains a link in G'. Define $K' = K \setminus \{(i, j)\}$.

If $l = i$, then $(b_k, k) = (b_i, a_i)$ or $(b_k, k) = (a_i, b_i)$ and the link (k, i) in G has been replaced by the link (k, j) in G'. In this case define $K' = K \setminus \{(i, j), (k, i)\} \cup \{(k, j)\}$. In those two subcases (1.5) remains valid.

In case 2 the link (i, j) is now a link in G'. Let (k, l) belong to K. Then $k < i$. If $l \neq j$, then (k, l) remains a link in G'. If $l = j$, then $(b_k, k) = (b_j, a_j)$ or $= (a_j, b_j)$, so that (k, i) is a link in G'. Define $K' = K \cup \{(i, j)\}$ in the first subcase and $K' = K \cup \{(i, j), (k, i)\} \setminus \{(k, j)\}$. Again (1.5) holds.

If case 3 holds, G has the form:

$$G = \left(\begin{array}{ccc} a_i & \dots & b_i \\ b_i & \dots & i \\ i & \dots & j \\ d_i & \dots & d_j \end{array}\right)$$

and eight subcases are to consider depending on whether a_i, b_i are equal or not to i, and d_i is equal to 0 or 2. The two cases $a_i = b_i = i$ can be dropped, for a is a permutation. The two cases $b_i \neq i$, $d_i = 2$ can also be dropped because of condition 4 for the matrices in $\mathcal{T}$. The case $a_i \neq i$, $b_i = i$, $d_i = 0$ is covered by case 1. There remain three subcases for which the mapping $G \mapsto G'$ is defined as follows:
Case 3′: $a_i \neq i$, $b_i \neq i$, $d_i = 0$.

$$G = \left(\begin{array}{ccc} a_i & \dots & b_i \\ b_i & \dots & i \\ i & \dots & j \\ 0 & \dots & d_j \end{array}\right) \mapsto \left(\begin{array}{ccc} b_i & \dots & a_i \\ a_i & \dots & i \\ i & \dots & j \\ 0 & \dots & d_j \end{array}\right) = G'.$$

Case $3''$: $a_i = i$, $b_i \neq i$, $d_i = 0$.

$$G = \begin{pmatrix} i & \dots & b_i \\ b_i & \dots & i \\ i & \dots & j \\ 0 & \dots & d_j \end{pmatrix} \mapsto \begin{pmatrix} b_i & \dots & i \\ i & \dots & i \\ i & \dots & j \\ 2 & \dots & d_j \end{pmatrix} = G'.$$

Case $3'''$: $a_i \neq i$, $b_i = i$, $d_i = 2$.

$$G = \begin{pmatrix} a_i & \dots & i \\ i & \dots & i \\ i & \dots & j \\ 2 & \dots & d_j \end{pmatrix} \mapsto \begin{pmatrix} i & \dots & a_i \\ a_i & \dots & i \\ i & \dots & j \\ 0 & \dots & d_j \end{pmatrix} = G'.$$

In those three subcases the pair (i, j) has remained a link in G'. Let (k, l) be a link in K different from (i, j). Then $k < i$. Also $l \neq j$. If $l \neq i$, then (k, l) remains a link in G'. If $l = i$, then $(b_k, k) = (b_i, a_i)$ or $(b_k, k) = (a_i, b_i)$ and the link (k, i) has been preserved in G'. We can then define: $K' = K$. Also (1.5) holds.

As for the proof of Capelli's identity we get $w(G', K') = -w(G, K)$ in cases 1, 2 and 3. Clearly, ω maps the first case to the second and conversely. Finally, subcase $3'$ goes to itself, and ω exchanges the two subcases $3''$ and $3'''$.

It follows that the sum of the weights of all the bad guys is zero, thus establishing (TUR).

3. What about the Anti-symmetric Analog? Howe and Umeda [H-U], and independently, Kostant and Sahi [K-S] discovered and proved an anti-symmetric analog of Capelli's identity. Although we, at present, are unable to give a combinatorial proof similar to the above proofs, we state this identity in the hope that one of our readers will supply such a proof. Since the anti-symmetric analog is only valid for even n, it is clear that the involution cannot be "local" as in the above involutions, but must be "global," i.e., involves many, if not all, matrices.

The Howe-Umeda-Kostant-Sahi Identity. *Let n be an even positive integer. Let $X = (x_{i,j})$ $(1 \leq i, j \leq n)$ be an anti-symmetric matrix: $x_{j,i} = -x_{i,j}$, and $P = (p_{i,j})$ be the corresponding anti-symmetric momenta matrix. Let*

$$(1'') \qquad A_{ij} := \sum_{k=1}^{n} x_{k,i} p_{k,j} + h(n - i - 1)\delta_{ij}.$$

Then

$$(\text{HU-KS}) \qquad \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) A_{\sigma 1,1} \dots A_{\sigma n,n} = \det X . \det P.$$

Although we are unable to prove the above identity combinatorially, we do know how to prove combinatorially another, less interesting, anti-symmetric analog of Capelli's identity, that is stated without proof at the end of Turnbull's paper [T].

Turnbull's Anti-Symmetric Analog. *Let $X = (x_{i,j})$ and $P = (p_{i,j})$ $(1 \leq i, j \leq n)$ be an anti-symmetric matrices as above. Let*

$$A_{ij} := \sum_{k=1}^{n} x_{k,i} p_{k,j} - h(n-i)\delta_{ij}, \tag{1''}$$

for $1 \leq i, j \leq n$. Then

$$\sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) A_{\sigma 1,1} \dots A_{\sigma n,n} = \operatorname{Per}(X^t P), \tag{TUR$'$}$$

where $\operatorname{Per}(A)$ denotes the permanent of a matrix A, and the matrix product $X^t P$ that appears on the right side of TUR$'$ is taken with the assumption that the $x_{i,j}$ and $p_{i,j}$ commute.

Since the proof of this last identity is very similar to the proof of Turnbull's symmetric analog (with a slight twist), we leave it as an instructive and pleasant exercise for the reader.

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