Alexander Burstein's Lovely Combinatorial Proof of John Noonan's Beautiful Theorem that the number of n-permutations that contain the Pattern 321 Exactly Once Equals (3/n)(2n)!/((n-3)!(n+3)!)

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Alex Burstein[1] gave a lovely combinatorial proof of John Noonan's[2] lovely theorem that the number of n-permutations that contain the pattern 321 exactly once equals $\frac{3}{n}\binom{2n}{n+3}$. Burstein's proof can be made even shorter as follows. Let $C_n := (2n)!/(n!(n+1)!)$ be the Catalan numbers. It is well-known (and easy to see) that $C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}$. It is also well-known (and fairly easy to see) that the number of 321-avoiding n-permutations equals C_n.

Any n-permutation, π, with exactly one 321 pattern can be written as $\pi_1 c \pi_2 b \pi_3 a \pi_4$, where cba is the unique 321 pattern (so, of course $a < b < c$). All the entries to the left of b, except c, must be smaller than b, and all the entries to the right of b, except for a, must be larger than b, or else another 321 pattern would emerge. Hence $\sigma_1 := \pi_1 b \pi_2 a$ is a 321-avoiding permutation of $\{1, \dots, b\}$ that does not end with b and $\sigma_2 := c\pi_3 b \pi_4$ is a 321-avoiding permutation of $\{b, \dots, n\}$ that does not start with b. This is a bijection between the Noonan set and the set of pairs (σ_1, σ_2) as above (for some $2 \leq b \leq n-1$). For any b the number of possible σ_1 is $C_b - C_{b-1}$. Similarly, the number of possible σ_2 is $C_{n-b+1} - C_{n-b}$. Hence the desired number is

$$\sum_{b=2}^{n-1} (C_b - C_{b-1})(C_{n-b+1} - C_{n-b}) = \sum_{b=1}^{n} (C_b - C_{b-1})(C_{n-b+1} - C_{n-b})$$

$$= \sum_{b=1}^{n} C_b C_{n-b+1} - \sum_{b=1}^{n} C_b C_{n-b} - \sum_{b=1}^{n} C_{b-1} C_{n-b+1} + \sum_{b=1}^{n} C_{b-1} C_{n-b}$$

$$= C_{n+2} - 2C_{n+1} - 2(C_{n+1} - C_n) + C_n = C_{n+2} - 4C_{n+1} + 3C_n = \frac{3}{n}\binom{2n}{n+3} \quad . \quad \Box$$

References

1. Alex Burstein, *A short proof for the number of permutations containing the pattern 321 exactly once*, Elec. J. Comb. **18(2)**(2011), #P21.

2. John Noonan, *The number of permutations containing exactly one increasing subsequence of length three*, Discrete Math. **152**(1996), 307-313 .

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