The Black-Scholes Equation WITHOUT Tears (or Probability!)

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The Fair price of a European Call Option (Black-Scholes Equation)

The Crown jewel of mathematical finance is

The Black-Scholes Equation([BS])

Suppose that a certain stock costs now S_0 dollars, its *volatility* is σ, and the *interest rate* is r. The *fair price* of an *option* to buy that stock, for the price of K dollars, at time T, is

$$S_0\,\Phi\left(\frac{\log\frac{S_0}{K}+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\;-\;K\,e^{-rT}\,\Phi\left(\frac{\log\frac{S_0}{K}+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\quad , \qquad (BS)$$

where

$$\Phi(z) := \int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \quad .$$

The purpose of the present, purely expository, *classroom note* is to derive it *from scratch*, **without any probability theory**, following the beautiful Cox-Ross-Rubinstein[CRR] discrete-time model, but in a way that should be understood by the proverbial smart middle-school student. The only prerequisite is the ability to solve two linear equations with two unknowns. I was greatly helped by Alison Etheridge's wonderful book[E].

Let S_T be the price of the stock at time T. Note that if $S_T < K$, it would be stupid for the buyer to *exercise* the option, since he can just buy the stock in the free market (rather than pay more for it), and he definitely lost the money spent buying the option. But if $S_T > K$, then he would exercise it, and by reselling it, would make $S_T - K$ dollars. If $S_T - K$ (adjusted) is larger than the price that he paid for the option then he came out ahead. Regarding the seller, he can "take a chance", pocket what he got from the buyer, and do nothing. If $S_T < K$ he lucked-out, but if $S_T > K$ then he would have to buy the stock for S_T dollars, and sell it to the buyer, at a loss, for K dollars, that is equivalent to paying the buyer $S_T - K$ dollars. If this (adjusted) amount is less than what he got for the option, then he still came out ahead, but if it is more, then he lost.

But doing nothing is only **one** option, (no pun intended!).

For the sake of simplicity, we will only consider the case where the interest rate, r, is zero. The extension to general r is straightforward.

Let's Bet!

A stock *today* costs 12 dollars. It is known that *tomorrow* it is either going to cost 6 dollars or 24 dollars.

The *seller* offers the *buyer*, if he pays him **8** dollars.

- 3 dollars if the stock would go down;

- 15 dollars if the stock would go up.

Schematically:

12

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6 [3] 24 [15]

Let's assume that the **seller does not buy any stock**. Then this deal is a **gamble** whether the stock would go up or down, and the actual prices, 6 and 24, are **irrelevant**.

If the stock would go down the buyer would lose 5 dollars (and the seller would make 5 dollars), but if the stock would go up, the buyer would make 7 dollars (and the seller would lose 7 dollars).

If the buyer estimates that the *probability* that the stock would go down is q, his expected gain is $q(-5) + (1-q)7 = 7 - 12q$, so he would agree to the deal if $q < \frac{7}{12}$, if he is **rational**.

But the *fair price* has **nothing to do** with what the buyer (or seller) think about the future of the stock. It is determined by the

AXIOM of OPTION PRICING: If you want to have a chance to make money, you should be willing to risk losing money! In other words:

IT SHOULD BE IMPOSSIBLE TO HAVE A STRATEGY THAT WOULD LET YOU MAKE MONEY (AT LEAST SOMETIMES) AND NEVER LOSE MONEY

It turns out that 8 dollars is too high of a price for the above deal.

Here is how to figure out the *fair price* of the above deal.

We assume that the seller of this bet has no money of his own, and will get in jail if he can not fulfill his obligations to the buyer. So he has to design a **mixed portfolio**, *only* using the money that he would get from the buyer of the bet.

Buy a units of stock and b dollars in cash (b can be negative, so he can borrow).

The value of this portfolio **today** is $12a + b$ (since one unit of stock costs 12 dollars today)

The value of this portfolio **tomorrow** is

- if the price of the stock went down to 6 dollars: $6\,a + b$

- if the price of the stock went up to 24 dollars: $24\,a + b$.

In order to *meet his obligations* (and not get arrested), he needs to make sure that

$$6a + b \geq 3 \quad , \quad 24a + b \geq 15 \qquad .$$

But if any of the inequalities is **strict** then he made money!, (at least sometimes). And that is not allowed, so to **barely** meet his obligations he has to solve the **system of two linear equations** in the **two unknowns**, a and b:

$$\{6a + b = 3 \quad , \quad 24a + b = 15\} \quad ,$$

whose solution is $a = \frac{2}{3}$, $b = -1$. The cost of this portfolio *today* is

$$12a + b = 12 \cdot \frac{2}{3} + (-1) = 7 \quad .$$

So the price must be at most 7 dollars. What if it is less? Then the *buyer* can *borrow* $\frac{2}{3}$ units of stock, and lend 1 dollar, and buy the deal, and come out ahead, forbidden!

So the **fair**, *arbitrage-free*, price of the bet is **exactly** seven dollars.

Note that the mixed portfolio 'buy $\frac{2}{3}$ units of stock and borrow 1 dollar', the so-called **hedging portfolio**, is **only** of theoretical interest. If the seller of the above deal would follow it, he is guaranteed to **never** make any money, so in real life he would use a different portfolio, based on his belief of the future market, and take a risk of losing money. But the price of seven dollars *guarantees* that neither buyer nor seller would be able to have a possibility of making money, without ever losing it, and hence is considered the *fair* price.

The General Single Period Binary Case

A stock *today* costs S_0 dollars. It is known that *tomorrow* it is either going to go down to S_d dollars or go up to S_u dollars.

The *seller* offers the *buyer*, the following deal.

- C_d dollars if the stock would go down;

- C_u dollars if the stock would go up.

Schematically:

S_0

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$S_d\,[C_d]$ $S_u\,[C_u]$

Here is how to figure out the *fair price* of that deal.

We assume that the seller has no money, and will get in jail if he can not fulfill his obligations. All he has is the money that he got from the buyer. How much money should the buyer give the seller for the above deal?

The seller has to design a **mixed portfolio**.

Buy a **units of stock and** b **dollars in cash** (b can be negative, so he can borrow).

The value of this portfolio **today** is $a \cdot S_0 + b$ (since one unit of stock costs S_0 dollars today).

The value of this portfolio **tomorrow** is

- if the price of the stock went down to S_d dollars: $a \cdot S_d + b$

- if the price of the stock went up to S_u dollars: $a \cdot S_u + b$.

In order to *meet his obligations* (and not get arrested), we need

$$aS_d + b \geq C_d \quad , \quad aS_u + b \geq C_u \quad .$$

But if any of the inequalities is **strict** then he would make money!, (at least sometimes), without any risk of losing money, and that is **forbidden**, so to **barely** meet his obligations he has to solve the **system of two linear equations and two unknowns**

$$aS_d + b \, = \, C_d \quad , \quad aS_u + b \, = \, C_u \quad ,$$

whose solution is

$$a = \frac{C_u - C_d}{S_u - S_d} \quad , \quad b = \frac{C_d S_u - C_u S_d}{S_u - S_d} \quad .$$

The cost of this portfolio *today* is

$$\frac{C_u - C_d}{S_u - S_d} \cdot S_0 \, + \, \frac{C_d S_u - C_u S_d}{S_u - S_d}$$

$$= \frac{(S_0 - S_d)\, C_u \, + \, (S_u - S_0)\, C_d}{S_u - S_d} \quad .$$

We just proved (using Algebra 1), the

Fundamental Lemma of Option Pricing

Suppose that the price of a certain stock today is S_0 dollars, and it is known that tomorrow it is either going to cost dS_0 ($d < 1$) or uS_0 ($u > 1$).

The fair price of a **deal** to be paid C_d dollars if the stock goes down and C_u dollars if the stock goes up is

$$\frac{(1-d)\, C_u \, + \, (u-1)\, C_d}{u - d} \, = \, \frac{1-d}{u-d}\, C_u \, + \, \frac{u-1}{u-d}\, C_d \quad .$$

Let's abbreviate

$$\alpha := \frac{1-d}{u-d} \quad , \quad \beta := \frac{u-1}{u-d} \quad .$$

Then the fair price is

$$\alpha C_u \, + \, \beta C_d \quad .$$

What if there are Two Days?

Suppose that today's stock costs S_0 dollars, and tomorrow it would go either down to S_d or up to S_u. If tomorrow it went down to S_d, then the day after tomorrow it may either go further down to $S_{d,d}$, or pick up, and go up to $S_{d,u}$. If tomorrow it went up to S_u, then the day after tomorrow it may drop to $S_{u,d}$ or go further up to $S_{u,u}$. Assume that the seller offers the buyer the following conditional promises

- $C_{d,d}$ dollars if after two days the price of the stock is $S_{d,d}$
- $C_{d,u}$ dollars if after two days the price of the stock is $S_{d,u}$
- $C_{u,d}$ dollars if after two days the price of the stock is $S_{u,d}$
- $C_{u,u}$ dollars if after two days the price of the stock is $S_{u,u}$

Then to figure out the fair price, one uses **backwards induction**. First figure out the fair price tomorrow under the assumption that the stock went down, get a certain number (using the fundamental lemma), let's call it C_d. Similarly for C_u, and then use the formula once again to get the fair price **today**.

Now let's make the simplifying assumption that if the price of the stock today is S, then the next day it is either dS or uS, and d and u are always the same from day to day. Hence

$$S_{d,d} = S_0 d^2 \quad , \quad S_{d,u} = S_{u,d} = S_0 ud \quad , \quad S_{u,u} = S_0 u^2 \quad .$$

Also let's assume that the payoff, at the end, only depends on the price of the stock, not on the history, so denoting $C_{u,u} = C_0, C_{u,d} = C_{d,u} = C_1, C_{d,d} = C_2$, it is immediate that the fair price, using the above formula is

$$\alpha(\alpha C_0 + \beta C_1) + \beta(\alpha C_1 + \beta C_2) \; = \; \alpha^2 C_0 + 2\alpha\beta C_1 + \beta^2 C_2 \quad .$$

If there are three days, then the price is

$$\alpha \cdot \left(\alpha^2 C_0 + 2\alpha\beta C_1 + \beta^2 C_2\right) + \beta \cdot \left(\alpha^2 C_1 + 2\alpha\beta C_2 + \beta^2 C_3\right)$$
$$= \alpha^3 C_0 + 3\alpha^2\beta C_1 + 3\alpha\beta^2 C_2 + \beta^3 C_3 \quad .$$

After n days,

$$\sum_{i=0}^{n} \binom{n}{i} \alpha^{n-i} \beta^i C_i \quad ,$$

that is easily proved by induction on n. Here C_i is the payoff if the price of the stock, after n days, is $S_0\, d^i u^{n-i}$.

So far it was abstract gambling. If the seller promises the buyer to sell the stock, after n days, for K dollars then the value of the promise is

$$C_i = S_0 d^i u^{n-i} - K \quad , \quad if \quad S_0 d^i u^{n-i} > K$$

$$C_i = 0 \quad , \quad if \quad S_0 d^i u^{n-i} < K \quad .$$

We just derived the

Discrete Black-Scholes Formula (Alias the Cox-Ross-Rubinstein Formula)

The fair price of the option is

$$\sum_{i; S_0 d^i u^{n-i} > K} \binom{n}{i} \alpha^{n-i} \beta^i \, (S_0 d^i u^{n-i} - K) \quad .$$

Defining (the *partial sum* of the binomial theorem)

$$B(n,m,p) := \sum_{i=0}^{m} \binom{n}{i} p^i (1-p)^{n-i} \quad ,$$

this can be expressed, more succinctly as

$$S_0 \cdot B(n,m,\beta d) - K \cdot B(n,m,\beta) \quad , \qquad (CCR)$$

where m is the solution of $S_0 d^m u^{n-m} = K$, and $\beta = \frac{u-1}{u-d}$.

Taking the limit

Going back to the 'continuous', Black-Scholes, scenario, if the expiration date is T, then chop it into n periods, where n is **very very big**. Each of the steps has duration

$$\Delta t = \frac{T}{n} \quad .$$

For a stock of 'volatility' σ, it turns out that, for very large n,

$$u = 1 + \sigma\sqrt{\Delta t} \quad , \quad d = 1 - \sigma\sqrt{\Delta t} \quad .$$

Computing $\beta := \frac{u-1}{u-d} = \frac{1}{2}$, plugging into (CCR), and taking the limit $n \to \infty$, using the fact, due to de Moivre (see [Z] for an elementary proof) that

$$B(n, np + \sqrt{np(1-p)}x) \to \Phi(x) \quad ,$$

with $p = \beta d = d/2$ for the first term, and with $p = \beta = \frac{1}{2}$ for the second term, of (CCR) one immediately gets (using Maple) the Black-Scholes equation (BS) (with $r = 0$) displayed at the beginning of this article.

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