Bijections for an identity of Young Tableaux

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Recall that *partition* $\lambda = (\lambda_1, \dots, \lambda_k)$ is a weakly-decreasing sequence of positive integers, and if $n = \lambda_1 + \cdots + \lambda_k$ we say that λ is a partition of n and write $\lambda \vdash n$ and $l(\lambda) = k$. For example $\lambda = (3, 3, 2, 2)$ is a partition of 10 and $l(\lambda) = 4$.

A convenient way to represent a partition λ is via its *Ferrers* diagram, that consists of $l(\lambda)$ left-justified lines of dots, such that the i-th row has λ_i dots. If you replace the dots by empty boxes, you would get what is called a *Young* diagram (of shape λ).

Finally recall that a *standard Young tableau* (SYT) of shape $\lambda \vdash n$ is any way of placing the integers $\{1, 2, \dots, n\}$ into the empty boxes of the Young diagram in such a way that all the rows and all the columns are increasing. For example $[[1, 2, 4, 6], [3, 5, 7], [8, 9]]$ is a SYT of shape $(4, 3, 2)$.

Let $H(k, \ell; n) = \{\lambda \vdash n \mid \lambda_{k+1} \leq \ell\}$ denote the partitions of n in the (k, ℓ) hook. For example, $H(k, 0; n)$ are the partitions of $\lambda \vdash n$ with $\ell(\lambda) \leq k$. Let f^λ denote the number of SYTs of shape λ. One then observes the following intriguing identity:

$$\sum_{\mu \in H(1,1;n+1)} (f^\mu)^2 = \sum_{\lambda \in H(2,0;2n)} f^\lambda.$$

We give this identity a bijective proof by showing that both

$$\sum_{\mu \in H(1,1;n+1)} (f^\mu)^2, \tag{1}$$

$$\sum_{\lambda \in H(2,0;2n)} f^\lambda, \tag{2}$$

can be mapped *bijectively* to the set of row-increasing matrices of shape (n, n) whose set of entries is $\{1, 2, \dots, 2n\}$, and hence, by composing, to each other. We describe these bijections. We leave it as pleasant excercises to the reader to formally prove that these are indeed bijections, by proposing inverse mappings, and proving that the compositions (in both direction) yield the identity mapping in each case.

The input for both (1) and (2) is a $2 \times n$ matrix of integers
$a_1 \dots a_n$
$b_1 \dots b_n$

such that $\{a_1, \dots, a_n\} \cup \{b_1, \dots b_n\} = \{1, 2, \dots, 2n\}$,

$a_1 < a_2 < ... < a_n$ and
$b_1 < b_2 < ... < b_n$.

Description of the bijection for (1)

Here the output is: Two standard tableaux of the same $(1,1)$-hook shape. Let $|\{a_1,\dots,a_n\}\cap\{1,\dots,n\}|=k$, so $\{a_1,\dots,a_n\}\cap\{1,\dots,n\}=\{a_{i_1}<\cdots<a_{i_k}\}$. Form now a SYT in the $(1,1)$ hook as follows:
Its (first) row is

$$1,\ \ a_{i_1}+1,\ \ \cdots\ ,a_{i_k}+1;$$

its (first) column is made of the remaining integers

$$\{1,\dots n+1\}\setminus\{\ 1,\ a_{i_1}+1,\ \ \cdots\ ,a_{i_k}+1\}=\{a'_{j_1},\dots,a'_{j_{n-k}}\}$$

in increasing order. This gives the first SYT – of $(1,1)$-hook shape $(k+1,1^{n-k})$.

It follows that

$$|\{a_1,\dots,a_n\}\cap\{n+1,\dots,2n\}|=n-k,$$

so denote

$$\{a_1,\dots,a_n\}\cap\{n+1,\dots,2n\}=\{b_{t_1}<\cdots<b_{t_{n-k}}\}.$$

Since $n+1\le b_{t_1}$, hence $2\le b_{t_1}-(n-1)$. Form the numbers

$$1<b_{t_1}-(n-1)<\cdots<b_{t_{n-k}}-(n-1)$$

and place them, in that (increasing) order, in the column of the second (1,1)-hook shape tableau, and the complement integers

$$\{1,\dots,n+1\}\setminus\{1,b_{t_1}-(n-1),\dots,b_{t_{n-k}}-(n-1)\}$$

in increasing order, in the row (after the corner 1). This gives the second SYT – again of shape $(k+1,1^{n-k})$. This map is clearly a bijection.

Example of bijection (1). Let $n=5$ and consider the 2×5 array

$$\begin{matrix}2, & 4, & 8, & 9, & 10\\ 1, & 3, & 5, & 6, & 7\end{matrix}$$

Now $\{2,\ 4,\ 8,\ 9,\ 10\}\cap\{1,\dots,5\}=\{2,4\}\rightarrow_{+1}\{3,5\}$ so the first row of the first tableau is $(1,3,5)$, hence $(n+1=6)$ its first column is $(1,2,4,6)^T$.

Also, $\{2,\ 4,\ 8,\ 9,\ 10\}\cap\{6,\dots,10\}=\{8,9,10\}\rightarrow_{-4}\{4,5,6\}$ so the first column of the second tableau is $(1,4,5,6)^T$, hence its first row is $(1,2,3)$. The pair of these two $(1,1)$-tableaux corresponds to the above array.

Description of the bijection for (2)

Here the output is a SYT whose shape is a $\le$ 2-rowed partition of $2n$ If for all i $a_i<b_i$, then it is a SYT, and do nothing (the output is the input).

Otherwise, let i be the smallest index such that $a_i>b_i$. Replace the above array by

$b_1 \; \dots \; b_{i-1} \;\; b_i \;\;\; a_i \;\;\; a_{i+1} \;\;\; \dots \;\;\; a_n$
$a_1 \; \dots \; a_{i-1} \; b_{i+1} \; b_{i+2} \; b_{i+3} \; \dots \; b_n$

If this is a SYT, then stop. Otherwise continue: Typically we arrive at a two rows array

$c_1, \; \dots \; c_s \; \dots \; c_r$
$d_1 \; \dots \; d_s$

with $s \le r$, $r + s = 2n$, with $c_1 < \cdots < c_r$ and with $d_1 < \cdots < d_s$. If $c_j < d_j$, $j = 1, \dots, s$ then this array is SYT and we are done. Otherwise let i be the smallest index such that $c_i > d_i$, then replace the above array by

$d_1 \; \dots \; d_{i-1} \;\; d_i \;\;\; c_i \;\;\; c_{i+1} \; \dots c_{s-1} \;\; \dots \;\; c_r$
$c_1 \; \dots \; c_{i-1} \;\; d_{i+1} \;\; d_{i+2} \; d_{i+3} \; \dots \; d_s$

Continue until a SYT is reached: thus the process stops at a SYT of shape $\lambda \in H(2,0;2n)$.

Example of bijection (2). Again consider the same 2×5 array as above, then the the bijection is as follows:

2, 4, 8, 9, 10
1, 3, 5, 6, 7 $\to$

1, 2, 4, 8, 9, 10
3, 5, 6, 7 $\to$

3, 5, 6, 7, 8, 9, 10
1, 2, 4 $\to$

1, 3, 5, 6, 7, 8, 9, 10
2, 4

Remark. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition, $\lambda_1 \ge \lambda_2 \ge \dots$ and denote $\lambda^{+1} = (\lambda_1 + 1, \lambda_2, \lambda_3 \dots)$. The same bijections can be applied to similar arrays of shape $(n+1, n)$, yielding a bijective proof for the SYT identity

$$\sum_{\mu \in H(2,0;2n+1)} f^{\mu} = \sum_{\lambda \in H(1,1;n+2)} f^{\lambda} \cdot f^{\lambda^{+1}}.$$

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