Efficient Evaluations of Weighted Sums over the Boolean Lattice inspired by conjectures of Berti, Corsi, Maspero, and Ventura

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Abstract: In their study of water waves, Massimiliano Berti, Livia Corsi, Alberto Maspero, and Paulo Ventura, came up with two intriguing conjectured identities involving certain weighted sums over the Boolean lattice. They were able to prove the first one, while the second is still open. In this methodological note, we will describe how to generate many terms of these types of weighted sums, and if in luck, evaluate them in closed-form. We were able to use this approach to give a new proof of their first conjecture, and while we failed to prove the second conjecture, we give overwhelming evidence for its veracity. In this second version, we are happy to announce that Mark van Hoeij was able to complete the proof of the second conjecture, by explicitly solving the second-order recurrence mentioned at the end.

An Intriguing Email message from Alberto Maspero

Awhile ago one of us (DZ) received an email message [M], with the following.

In our current study of water waves [BCMV], continuing our work in [BMV], and other papers, we came across the following sums.

Let $p \geq 2$, $1 \leq q \leq p-1$ and $0 < j_1 < j_2 < \ldots < j_q < p$ be positive integers. Define

$$n_q^{(p)}(j_1, \ldots, j_q) := (-12)^q j_1 \cdots j_q \, \frac{j_1(j_2-j_1)\cdots(j_q-j_{q-1})(p-j_q)}{(p^3-p+j_1-j_1^3)\cdots(p^3-p+j_q-j_q^3)} \quad , \tag{1}$$

and

$$g_q^{(p)}(j_1, \ldots, j_q) := -\frac{28}{9}p^2 + \frac{49}{45}q + \frac{32}{9}j_1^2 - \frac{4}{9} - \frac{4}{9}\frac{p^3-p}{j_1} + \frac{5}{18}(p^3-p)\left(\frac{1}{j_1} + \cdots + \frac{1}{j_q}\right) \tag{2}$$

$$\frac{38}{15}\left(j_1^2 + \cdots + j_q^2\right) + \frac{p^3-p}{5}\left(\frac{p+j_1}{p^2+j_1^2+pj_1-1} + \ldots + \frac{p+j_q}{p^2+j_q^2+pj_q-1}\right) - \frac{13}{9}(j_1j_2 + j_2j_3 + \cdots + j_qp) \quad .$$

The following identities seem to hold.

$$\sum_{q=1}^{p-1} \sum_{0<j_1<\ldots<j_q<p} n_q^{(p)}(j_1, \ldots, j_q) \; = \; -p \quad . \tag{3}$$

$$\sum_{q=1}^{p-1} \sum_{0<j_1<\ldots<j_q<p} n_q^{(p)}(j_1, \ldots, j_q) \, g_q^{(p)}(j_1, \ldots . j_q) \; = \; p(p+1)^2 \quad . \tag{4}$$

Maspero concluded:

"We actually managed, after dire efforts, to prove (3)*, whereas claim* (4) *seems out of reach. We just verified it for* $p = 2, \ldots, 21$.*"*

Why are such sums interesting?

Note that the sums in (3) and (4) are *weighted sums* defined over all non-empty subsets of the set $\{1...p-1\}$. Hence to compute many terms, straight from the definition, is exponentially expensive.

They are a bit unnatural in that the symbol p has both the roles of *integer* and of *variable*. Our first step is to *decouple* these two roles of p, introduce a *formal* variable x, and define

$$N(x;j_1,\dots,j_q) := (-12)^q j_1\cdots j_q \, \frac{j_1(j_2-j_1)\cdots(j_q-j_{q-1})(x-j_q)}{(x^3-x+j_1-j_1^3)\cdots(x^3-x+j_q-j_q^3)} \quad , \tag{1'}$$

$$G(x;j_1,\dots,j_q) := -\frac{28}{9}x^2+\frac{49}{45}q+\frac{32}{9}j_1^2-\frac{4}{9}-\frac{4}{9}\frac{x^3-x}{j_1}+\frac{5}{18}(x^3-x)\left(\frac{1}{j_1}+\cdots+\frac{1}{j_q}\right) \tag{2'}$$

$$\frac{38}{15}\left(j_1^2+\cdots+j_q^2\right)+\frac{x^3-x}{5}\left(\frac{x+j_1}{x^2+j_1^2+xj_1-1}+\ldots+\frac{x+j_q}{x^2+j_q^2+xj_q-1}\right)-\frac{13}{9}(j_1j_2+j_2j_3+\cdots+j_{q-1}j_q+j_qx) \quad .$$

Note that $N(p;j_1,\dots,j_q)=n_q^{(p)}(j_1,\dots,j_q)$ and $G(p;j_1,\dots,j_q)=g_q^{(p)}(j_1,\dots,j_q)$.

We are interested in efficient computation, and if possible, explicit evaluation of

$$A_p(x) := \sum_{q=1}^{p-1}\sum_{0<j_1<\ldots<j_q<p} N(x;j_1,\dots,j_q) \quad , \tag{3'}$$

and

$$C_p(x) := \sum_{q=1}^{p-1}\sum_{0<j_1<\ldots<j_q<p} N(x;j_1,\dots .j_q)G(x;j_1,\dots .j_q) \quad . \tag{4'}$$

In order to facilitate *dynamical programming*, it is natural to consider these weighted sums where the largest member of the subset, j_q, is fixed. So we define

$$B_p(x) := \sum_{0<j_1<\ldots<j_q=p} N(x;j_1,\dots,j_q) \quad , \tag{3''}$$

and

$$D_p(x) := \sum_{0<j_1<\ldots<j_q=p} N(x;j_1,\dots,j_q)G(x;j_1,\dots,j_q) \quad . \tag{4''}$$

Once the quantities $B_p(x)$ and $D_p(x)$ are known, our original quantities of interest, $A_p(x)$ and $C_p(x)$, can be evaluated using

$$A_p(x) = \sum_{p'=1}^{p-1} B_{p'}(x) \quad .$$

$$C_p(x) = \sum_{p'=1}^{p-1} D_{p'}(x) \quad .$$

The general framework

Note that the weights $N(x; j_1, \ldots, j_q)$ and $G(x; j_1, \ldots, j_q)$ have a *recursive "Markovian" structure.*

• If you know $N(x; j_1, \ldots, j_{q-1})$, you can quickly get $N(x; j_1, \ldots, j_{q-1}, j_q)$, by **multiplying** by a certain function of (j_{q-1}, j_q).

• If you know $G(x; j_1, \ldots, j_{q-1})$, you can quickly get $G(x; j_1, \ldots, j_{q-1}, j_q)$, by **adding** a certain (different) function of (j_{q-1}, j_q).

This leads us to consider the following general set-up.

Definition: Let $f_1(X)$ be an arbitrary uni-variate function, and $f_2(X, Y)$ an arbitrary bivariate function. Define the weight, for singleton sets $\{j_1\}$

$$W(f_1, f_2; [j_1]) := f_1(j_1) \quad ,$$

and for sets with more than one element, recursively (where we write $j_1, \ldots, j_q$ in increasing order):

$$W(f_1, f_2; [j_1, \ldots, j_q]) := W(f_1, f_2; [j_1, \ldots, j_{q-1}]) \cdot f_2(j_{q-1}, j_q) \quad .$$

Similarly let $g_1(X)$ and $g_2(X, Y)$ be arbitrary univariate and bivariate functions and define

$$V(g_1, g_2; [j_1]) := g_1(j_1) \quad ,$$

and for sets with more than one element, recursively

$$V(g_1, g_2; [j_1, \ldots, j_q]) := V(g_1, g_2; [j_1, \ldots, j_{q-1}]) + g_2(j_{q-1}, j_q) \quad .$$

Note that the original [BCMV] summations have the following f_1, f_2, g_1, g_2:

$$f_1(X) = -\frac{12X^2 (x - X)}{-X^3 + x^3 + X - x} \quad ,$$

$$f_2(X, Y) = -\frac{12Y (Y - X)(x - Y)}{(x - X)(-Y^3 + x^3 + Y - x)} \quad ,$$

$$g_1(X) = -\frac{28x^2}{9} + \frac{29}{45} + \frac{274X^2}{45} - \frac{x^3 - x}{6X} + \frac{(x^3 - x)(x + X)}{5X^2 + 5xX + 5x^2 - 5} - \frac{13xX}{9} \quad ,$$

$$g_2(X, Y) = \frac{\frac{5}{18}x^3 - \frac{5}{18}x}{Y} + \frac{38Y^2}{15} + \frac{(x^3 - x)(x + Y)}{5Y^2 + 5xY + 5x^2 - 5} - \frac{13XY}{9} - \frac{13(Y - X)x}{9} + \frac{49}{45} \quad .$$

The general analogs of $A_p(x)$, $B_p(x)$, $C_p(x)$ and $D_p(x)$, let's call them a_p, b_p, c_p, and d_p, respectively: are

$$b_p := \sum_{0 < j_1 < \ldots < j_q = p} W(f_1, f_2; [j_1, \ldots, j_q]) \quad ,$$

and then

$$a_p = \sum_{p'=1}^{p-1} b_{p'} \quad .$$

$$d_p := \sum_{0<j_1<\ldots<j_q=p} W(f_1,f_2;[j_1,\ldots,j_q])\, V(f_1,f_2;[j_1,\ldots,j_q]) \quad ,$$

and then

$$c_p = \sum_{p'=1}^{p-1} d_{p'} \quad .$$

Let's first try to examine b_p.

We can break-up the sum that defines b_p, where every summand has $j_q = p$, according to the value of j_{q-1}:

$$b_p := \sum_{\substack{0<j_1<\ldots<j_q \\ j_q=p}} W(f_1,f_2;[j_1,\ldots,j_q]) \;=\; \sum_{p'=1}^{p-1} \sum_{\substack{0<j_1<\ldots<j_{q-1}<p \\ j_{q-1}=p'}} W(f_1,f_2;[j_1,\ldots,j_{q-1},p]) \;=$$

$$\sum_{p'=1}^{p-1} \sum_{\substack{0<j_1<\ldots<j_{q-1} \\ j_{q-1}=p'}} W(f_1,f_2;[j_1,\ldots,j_{q-1}]) \cdot f_2(p',p) \;=$$

$$\sum_{p'=1}^{p-1} f_2(p',p) \left(\sum_{0<j_1<\ldots<j_{q-1}=p'} W(f_1,f_2;[j_1,\ldots,j_{q-1}]) \right) \;=\; \sum_{p'=1}^{p-1} f_2(p',p)\, b_{p'} \quad .$$

Hence the sequence b_p can be computed in quadratic-time using the recurrence

$$b_p \;=\; \sum_{p'=1}^{p-1} f_2(p',p)\, b_{p'} \quad ,$$

subject to the *initial condition*

$$b_1 = f_1(1) \quad .$$

A similar argument, that we omit, enables us to get a quadratic-time recurrence for d_p, that assumes that b_p is already known.

Once we get a hold of b_p and d_p, we can recover a_p and c_p using $a_p = \sum_{p'=1}^{p-1} b_{p'}$ and $c_p = \sum_{p'=1}^{p-1} d_{p'}$.

Let's go back to specializing to the [BCMV] (already proved by them) conjecture (3).

If we are lucky and we can conjecture an explicit expression for $B_p(x)$, then all we have to do is verify that this conjectured expression also satisfies the same recurrence and initial condition. With the above $f_2(X,Y)$ the recurrence becomes

$$B_p(x) \; = \; -12 \frac{p(x-p)}{x^3-x+p-p^3} \left(p + \sum_{p'=1}^{p-1} \frac{p-p'}{x-p'} B_{p'}(x) \right) \quad , \tag{5}$$

with the initial condition $B_0(x) = 0$.

Cranking out the first 20 terms one easily conjectures

$$B_p(x) \; = \; \frac{12p^2\,(p-x)}{x\,(x+1)\,(x-1)} \quad ,$$

and it is routine to verify (even by hand, but Maple is glad to do it for you) that (5) is satisfied if $B_p(x)$ is replaced by the above right side. Then we ask Maple to kindly sum

$$A_p(x) = \sum_{p'=1}^{p-1} \frac{12p'^2\,(p'-x)}{x\,(x+1)\,(x-1)} \quad ,$$

giving

$$A_p(x) = \frac{p\,(p-1)\left(3\,p^2 - 4\,xp - 3\,p + 2\,x\right)}{x\,(x-1)\,(x+1)} \quad .$$

Now what [BCMV] are really interested in is not $A_p(x)$, in general, but the special case $x = p$, i.e. in $A_p(p)$. Plugging-in $x = p$ above, and simplifying gives that indeed

$$A_p(p) \; = \; -p \quad .$$

So we have a new proof of the already-proved-by-them identity (3) of [BCMV].

We can get similar dynamical programming (quadratic-time) recurrence for $D_p(x)$ that expresses it in terms of previous values $\{D_{p'}(x) : 1 \leq p' \leq p-1\}$ **and** (the already known) $B_p(x)$. Using the above *proved* expression for the latter, we can compute many terms. Alas, it is no longer a nice rational function, and the sequence seems very complicated. But using the *holonomic ansatz* [Z] (see [K] for a great Mathematica implementation) one can first *guess* (very complicated!) linear recurrences for both $D_p(x)$ and $C_p(x)$, that nevertheless, to our pleasant surprise, are mere second order (but with very complicated coefficients). See procedures `DxH(p,x)` and `CxH(p,x)` in our Maple package mentioned below. These recurrences are first guessed, using *undetermined coefficients*, implemented in our Maple package `FindRec.txt`, available from:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/FindRec.txt` .

Once guessed, they are all automatically and *rigorously* provable using the holonomic ansatz as implemented by Koutschan, i.e. the sequences defined by these second-order recurrences also satisfy the original recurrences.

This enables us to easily compute the first 2000 terms of the sequence $\{C_p(x)\}$ that are all very complicated rational functions of x. But when we plug-in $x = p$ the sequence $\{C_p(p)\}$ **coincides** with the conjectured sequence $\{p(p+1)^2\}$. Of course, this is not a rigorous proof, but being *empiricists*, knowing that it is true for the first 2000 terms, is **good enough for us**.

Maple package and input and output files

Everything is implemented in the Maple package `BCMV.txt` available from

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/BCMV.txt` .

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`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/bcmv.html` ,

contains the input and output files that *rigorously* prove the above explicit expressions for $A_p(x)$ and $B_p(x)$, and that empirically verifies (4) all the way to $p = 2000$.

Postscript written Feb. 28, 2024: Mark van Hoeij met our challenge, to explicitly solve the recurrence satisfied by $C_p(x)$, that enables plugging-in $x = p$ into it and proving that indeed $C_p(p) = p(p+1)^2$.

See:

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/bcmvChallenge.txt` .

For a detailed explanation, see the postscript kindly written by Mark van Hoeij:

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/bcmvMvH.html` .

This completes the (rigorous!) proof of Conjecture (4). A donation of 100 dollars to the OEIS, in Mark van Hoeij's honor, has been made.

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