Explicit (Polynomial!) Expressions for the Expectation, Variance and Higher Moments of the Size of a $(2n+1, 2n+3)$-core partition with Distinct Parts

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Abstract

Armin Straub's beautiful article (`https://arxiv.org/abs/1601.07161`) concludes with two intriguing conjectures about the number, and maximal size, of $(2n+1, 2n+3)$-core partitions with distinct parts. These were proved by ingenious, but complicated, arguments by Sherry H.F. Yan, Guizhi Qin, Zemin Jin, and Robin D.P. Zhou (`https://arxiv.org/abs/1604.03729`). In the present article, we first comment that these results can be proved faster by "experimental mathematics" methods, that are easily rigorizable. We then develop relatively efficient, symbolic-computational, algorithms, based on non-linear functional recurrences, to generate what we call the Straub polynomials, where $S_n(q)$ is the generating function, according to size, of the set of $(2n+1, 2n+3)$-core partitions with distinct parts, and compute the first 21 of them. These are used to deduce explicit expressions, as polynomials in n, for the mean, variance, and the third through the seventh moments (about the mean) of the random variable "size" defined on $(2n+1, 2n+3)$-core partitions with distinct parts. In particular we show that this random variable is not asymptotically normal, and the limit of the coefficient of variation is $\sqrt{14010}/150 = 0.789092305...$, the scaled-limit of the third moment (skewness) is $(396793/390815488) \cdot \sqrt{467 \cdot 7680} = 1.92278748..$, and that the scaled-limit of the 4th-moment (kurtosis)

is 145309380/16792853 = 8.6530490.... We are offering to donate one hundred dollars to the OEIS foundation in honor of the first to identify the limiting distribution.

Supporting Maple packages and output

All the results in this article were obtained by the use of the Maple packages

- `http://www.math.rutgers.edu/~zeilberg/tokhniot/Armin.txt`,
- `http://www.math.rutgers.edu/~zeilberg/tokhniot/core.txt`,

whose output files, along with links to diagrams, are available from the *front* of this article

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/armin.html`.

1 (s, t)-core partitions and Drew Armstrong's ex-conjecture

Recall that a *partition* is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $k \geq 0$, called its *number of parts*; $n := \lambda_1 + \cdots + \lambda_k$ is called its *size*, and we say that λ is a *partition of n*. Also recall that the *Ferrers diagram* (or equivalently, using empty squares rather than dots, *Young diagram*) of a partition λ is obtained by placing, in a *left-justified* way, λ_i dots at the i-th row. For example, the Ferrers diagram of the partition

$(5, 4, 2, 1, 1)$ is

$$\begin{array}{ccccc} * & * & * & * & * \\ * & * & * & * & \\ * & * & & & \\ * & & & & \\ * & & & & \end{array} .$$

Recall also that the *hook length* of a dot (i, j) in the Ferrers diagram, $1 \leq j \leq \lambda_i$, is the number of dots to its right (in the same row) plus the number of dots below it (in the same column) plus one (for itself), in other words $\lambda_i - i + \lambda'_j - j + 1$, where λ' is the *conjugate partition*, obtained by reversing the roles of rows and columns. (For example if $\lambda = (5, 4, 2, 1, 1)$ as above, then $\lambda' = (5, 3, 2, 2, 1)$).

Here is a table of hook-lengths of the above partition, $(5, 4, 2, 1, 1)$:

$$\begin{array}{ccccc} 9 & 6 & 4 & 3 & 1 \\ 7 & 4 & 2 & 1 & \\ 4 & 1 & & & \\ 2 & & & & \\ 1 & & & & \end{array} .$$

It follows that its set of hook-lengths is $\{1, 2, 3, 4, 6, 7, 9\}$. A partition is called an s-core if none of its hook-lengths is s. For example, the above partition, $(5, 4, 2, 1, 1)$, is a 5-core, and an i-core for all $i \geq 10$.

A partition is a *simultaneous* (s, t)-core partition if it avoids both s and t. For example the above partition, $(5, 4, 2, 1, 1)$, is a $(5, 11)$-core partition (and a $(5, 12)$-core partition, and a $(100, 103)$-core partition etc.).

For a lucid and engaging account, see [AHJ].

As mentioned in [AHJ], Jaclyn Anderson ([A]) very elegantly proved the following.

Theorem ([A]): If s and t are relatively prime positive integers, then there are **exactly**

$$\frac{(s + t - 1)!}{s!t!},$$

(s, t)-core partitions.

For example, here are the $(3+5-1)!/(3!5!) = 7$ $(3,5)$-core partitions:

$$\{empty, 1, 2, 11, 31, 211, 4211\}.$$

Drew Armstrong ([AHJ], conjecture 2.6) conjectured, what is now the following theorem.

Theorem ([J]) : The *average size* of an (s,t)-core partition is given by the nice polynomial

$$\frac{(s-1)(t-1)(s+t+1)}{24}.$$

For example, the (respective) sizes of the above-mentioned $(3,5)$-core partitions are

$$0, 1, 2, 2, 4, 4, 8;$$

hence, the average size is

$$\frac{0+1+2+2+4+4+8}{7} = \frac{21}{7} = 3,$$

and this agrees with Armstrong's conjecture, since

$$\frac{(3-1)(5-1)(3+5+1)}{24} = 3.$$

Armstrong's conjecture was proved by Paul Johnson ([J]) using a very complicated (but ingenious!) argument (that does much more). Shortly after, and almost *simultaneously* (no pun intended) it was re-proved by Victor Wang [Wan], using another ingenious (and even more complicated) argument, that also does much more, in particular, proving an intriguing conjecture of Tewodros Amdeberhan and Emily Sergel ([AL]). Prior to the full proofs by Johnson and Wang, Richard Stanley and Fabrizio Zanello [StaZ] came up with a nice (but rather *ad hoc*) proof of the important special case of $(s, s+1)$-core partitions. An explicit expression for the *variance* was found by Marko Thiel and Nathan Williams ([TW]).

Ekhad and Zeilberger ([EZ]) went far beyond, and derived explicit expressions for the first 6 moments for the general (s,t)-core partitions, and the first 9 moments for the case $(s, s+1)$, and used them to find the scaled limits up to the ninth, that strongly suggest that the limiting distribution is the continuous random variable

$$\sum_{k=1}^{\infty} \frac{z_k^2 + \tilde{z}_k^2}{4\pi^2 k^2},$$

where z_k and $\tilde{z}_k$ are jointly independent sequences of independent standard normal random variables.

2 Simultaneous core partitions into distinct parts

Tewodros Amdeberhan ([Am]) initiated the study of simultaneous core partitions with *distinct parts*, and conjectured that the number of $(s, s+1)$-core partitions with distinct parts is given by the Fibonacci number F_{s+1}. This was proved by Armin Straub ([Str]) and Huan Xion ([X]). Xion also proved a conjectured expression of Amdeberhan for the expected size, in terms of a double sum involving Fibonacci numbers. A more explicit expression was derived by the first-named author [Za], who also derived, assisted by his computer, **explicit expressions** (as rational functions in F_s, F_{s+1}, and s) for the first 16 moments. He then deduced that the scaled moments tend to the moments of the standard normal distribution, giving strong evidence (that could be turned into a fully rigorous proof, using the method of [Ze2]) that the random variable 'size' defined over *distinct* $(s, s+1)$-core partitions is *asymptotically normal*.

This is surprising, since, as already mentioned above, it was shown in [EZ] that when defined over all (not necessarily distinct) partitions, the random variable 'size' is **not** asymptotically normal.

At the end of his beautiful paper, [Str], (where, among many other things, the author describes a beautiful new elegant partition identity between Odd and Distinct integer partitions which preserves the perimeter, that should have been found by Euler (but had to wait for Straub)) Armin Straub conjectured two intriguing enumeration results.

Theorem 0 (conjectured in [Str], first proved in [YQJZ]) The *number* of $(2n+1, 2n+3)$-core partitions with distinct parts equals 4^n.

Theorem 0': (conjectured in [Str], first proved in [YQJZ]) The largest size of a $(2n{+}1, 2n{+}3)$-core partition with distinct parts is $\frac{1}{24}\,(5\,n+11)\,n\,(n+2)\,(n+1)$.

The proofs in [YQJZ] use ingenious, but rather complicated, combinatorial arguments. We will, in this article, give new, much simpler, 'experimental-mathematical' proofs, that can be easily made rigorous. But our main purpose is to establish *explicit* expressions for the expectation, variance, and all the moments up to the seventh. With more computing power, it should be possible to go beyond. We then go on and use these explicit (polynomial) expressions in order to find the limits of the scaled moments, giving exact values for the first seven moments of the limiting (scaled) probability distribution of the random variable 'size' over $(2n+1, 2n+3)$-core partitions with distinct parts (as $n \to \infty$), and one of us (DZ) is pledging $100 to the OEIS foundation for identifying that limiting (continuous) probability distribution.

2.1 Explicit expressions for the first seven moments

Theorem 1: The average size of a $(2n{+}1, 2n{+}3)$-core partition with distinct parts is

$$\frac{1}{32}(10\,n^3 + 27\,n^2 + 19\,n).$$

Note that the corresponding average taken over *all* partitions, according to Armstrong's ex-conjecture, is $\frac{1}{6}n(n+1)(2n+5) = \frac{1}{3}n^3 + O(n^2)$, while, according to Theorem 1, our average (i.e. for the distinct case) is $\frac{5}{16}n^3 + O(n^2)$, so it is a bit less.

Theorem 2: The variance of the random variable 'size' defined on the set of $(2n+1, 2n+3)$-core partitions with distinct parts is

$$\frac{1}{15360}(934\,n^6 + 4687\,n^5 + 9700\,n^4 + 10505\,n^3 + 6256\,n^2 + 1518\,n).$$

Note that according to [EZ], the corresponding variance, taken over *all* partitions is

$$\frac{1}{720}\,(2\,n+1)\,(2\,n+3)\,(2\,n+2)\,n\,(4\,n+5)\,(4\,n+4)\,,$$

which is $\frac{8}{45}n^6+O(n^5) = 0.1777777778n^6+O(n^5)$, while for our case, according to Theorem 2, it is $\frac{467}{7680}n^6 + O(n^5) = 0.06080729167n^6 + O(n^5)$.

Theorem 3: The third moment (about the mean) of the random variable 'size' defined on $(2n+1, 2n+3)$-core partitions with distinct parts is

$$\frac{1}{27525120} \cdot (793586\, n^9 + 4945025\, n^8 + 12775144\, n^7 + 17215282\, n^6 + 11839450\, n^5 + 1535905\, n^4 - 4756804\, n^3 - 4342612\, n^2 - 1297776\, n).$$

Theorem 4: The fourth moment (about the mean) of the random variable 'size' defined on $(2n+1, 2n+3)$-core partitions with distinct parts is

$$\frac{1}{54499737600} \cdot (1743712560\, n^{12} + 13490284234\, n^{11} + 45408125279\, n^{10} + 87568584895\, n^9 + 109173019890\, n^8 + 97494786972\, n^7 + 68082466947\, n^6 + 34594762895\, n^5 + 8734303600\, n^4 + 3269131844\, n^3 + 7648567524\, n^2 + 4135638960\, n).$$

Theorem 5: The fifth moment (about the mean) of the random variable 'size' defined on $(2n+1, 2n+3)$-core partitions with distinct parts is

$$\frac{1}{108825076039680} \cdot n\,(n+1)\,(4115597238066\, n^{13} + 30331407775461\, n^{12} + 93240357590320\, n^{11} + 153901186416765\, n^{10} + 154511084293844\, n^9 + 126787455814599\, n^8 + 115227024155664\, n^7 + 42586120680111\, n^6 - 95604599727502\, n^5 - 105409116317640\, n^4 + 43165327777096\, n^3 + 91113907956144\, n^2 - 30975685518528\, n - 65049004454400).$$

Theorem 6: The sixth moment (about the mean) of the random variable 'size' defined on $(2n+1, 2n+3)$-core partitions with distinct parts is

$$\frac{1}{8288117791182028800} \cdot$$

$$(459077029253573970\, n^{18} + 3986958940758529155\, n^{17} + 14588638597341766281\, n^{16} + 29315654117562943844\, n^{15} + 38855616058049391120\, n^{14} + 52048632801161949890\, n^{13}$$

$$+87053992212835094382\,n^{12}+102228197171521441748\,n^{11}+24538654588404043230\,n^{10}$$
$$-8106339791824458684\,5n^{9}-37681424022539337807\,n^{8}+128753068232342353072\,n^{7}$$
$$+136357236921377110920\,n^{6}-109095423240535042640\,n^{5}-26455556672455622385\,6n^{4}$$
$$-62480060539123323264\,n^{3}+164786511770490504960\,n^{2}+100625844884387235840\,n)\ .$$

Theorem 7: The seventh moment (about the mean) of the random variable 'size' defined on $(2n+1, 2n+3)$-core partitions with distinct parts is

$$\frac{n(n+1)}{2^{40}\cdot 3^{5}\cdot 5^{2}\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19}\cdot$$

$$(20325334435585878483 0\,n^{19}+152594151827767306263 5\,n^{18}$$
$$+437609078089003231069 4\,n^{17}+592053224482703695472 4\,n^{16}$$
$$+710818114733299438159 8\,n^{15}+225166148626190416574 40\,n^{14}$$
$$+477377544325424687507 10\,n^{13}+214315381833860521913 06\,n^{12}$$
$$-771273497909452212216 52\,n^{11}-987886085309446797821 07\,n^{10}$$
$$+914686281751886999007 48\,n^{9}+276198594921821905993 026\,n^{8}$$
$$+164310592679893652073 504\,n^{4}+142083751440080403128 1984\,n^{3}$$
$$+531526793585839194753 60\,n^{7}-516374679437475960870 016\,n^{6}$$
$$-696941224296942655687 312\,n^{5}+110998519763030897571 5328\,n^{2}$$
$$-745951061503715454673 920\,n-102638755126984928882 6880).$$

2.2 Corollaries

1. The limit of the "coefficient of variation", as $n \to \infty$, is $\frac{1}{150}\sqrt{14010} = 0.7890923055426827989\ldots$. In particular, unlike $(k, k+1)$-core partitions with distinct parts discussed in [Za], there is **no** concentration about the mean.

2. The limit of the *skewness*, as $n \to \infty$, is $\frac{396793}{390815488}\sqrt{467}\sqrt{7680}$ $= 1.9227874808883586667\ldots$

3. The limit of the *kurtosis*, as $n \to \infty$, is $\frac{145309380}{16792853} = 8.6530490084085\ldots$

4. The limit of the scaled fifth moment (α_5), as $n \to \infty$, is $\frac{3429664365055}{156594294624768}\sqrt{467}\sqrt{7680} = 41.4777067204457\ldots$

5. The limit of the scaled sixth moment (α_6), as $n \to \infty$, is $\frac{382564191044644975}{1552893421695616} = 246.35572905\ldots.$

6. The limit of the scaled seventh moment (α_7), as $n \to \infty$, is $\frac{56459262321071884675}{6298890665465234636 8}\sqrt{467}\sqrt{7680} = 697.5015509357\ldots$

3 Proving the theorems

We now explain the methods used to obtain the results in the previous section.

3.1 A New ("Experimental Math") proof of Armin Straub's ex-conjecture that the number of $(2n+1, 2n+3)$-core partitions with distinct parts equals 4^n

The way Jaclyn Anderson proved her celebrated theorem ([An]) that if $gcd(s,t) = 1$, then the number of (s,t)-core partitions equals $(s+t-1)!/(s!t!)$ was by defining a bijection with the set of **order ideals** of the poset

$$P_{s,t} := \mathbb{N} \backslash (s\mathbb{N} + t\mathbb{N}),$$

where $\mathbb{N} = \{0, 1, 2, 3, \ldots,\}$ is the set of non-negative integers, and the partial-order relation $c \leq_P d$ holds whenever $d - c$ can be expressed as $\alpha s + \beta t$ for some $\alpha, \beta \in \mathbb{N}$.

The set of order ideals of $P_{s,t}$, in turn, is in bijection with the set of *lattice paths* in the two-dimensional square lattice, from $(0,0)$ to (s,t) lying above the line $sy - tx = 0$. This correspondence is used in the Maple package `core.txt`, and was used in [Za], but for our present purposes it is more efficient to use order ideals.

Recall that an order ideal I, in a poset P, is a set of vertices of P such that if $c \in I$ then all elements, d, such that $d \leq_P c$ also belong to I. Equivalently, if

d does **not** belong to I, then all vertices c 'above' it (i.e. such that $c \geq_P d$), also do **not** belong to I.

Let $s(n)$ be the number of order ideals of the lattice $P_{2n+1,2n+3}$ with no consecutive labels. Recall that, thanks to Jaclyn Anderson, this is the number of $(2n+1, 2n+3)$-core partitions with distinct parts, our *object of desire*.

Let's try and find an algorithm to compute the sequence $\{s(n)\}$ for as many terms as possible.

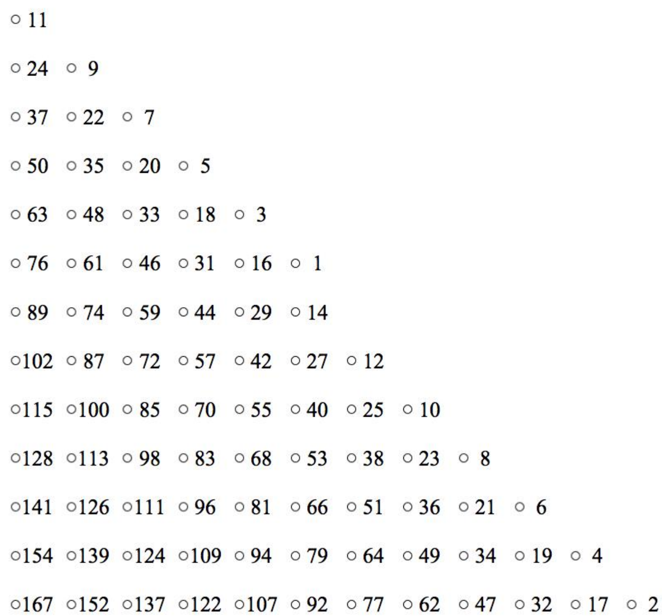
Let's review first how to prove that the number of order ideals of $P_{k+1,k+2}$, let's call it $p(k)$, is the Catalan number C_{k+1}. Let i be the smallest empty label on the hypotenuse, implying that $1, \dots, i-1$ are occupied, and 'kicking out' all vertices that are $\geq_P$ of the vertex labeled i, leaving us with two connected components, triangles of sizes $i-2$ and $k-i$, with independent decisions regarding their order ideals. The 'initial conditions' are $p(-1) = 1$, $p(0) = 1$, and for $k \geq 1$, we have

$$p(k) = \sum_{i=1}^{k+1} p(i-2)p(k-i). \tag{0}$$

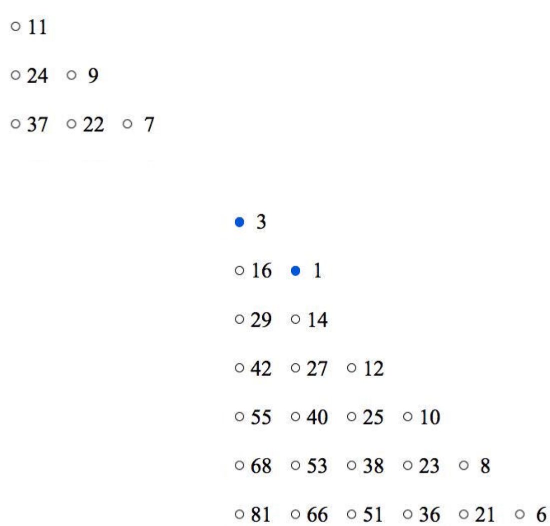
Now let's move-on to finding $s(n)$, i.e. the number of order ideals of $P_{2n+1,2n+3}$ without consecutive labels.

A diagram of the lattice $P_{2n+1,2n+3}$ (for $n = 6$) can be found in Figure 1(a) (see also Figure 3 (page 5) of [YQJZ], where the lattice is drawn such that the rank-zero vertices are at the bottom rather than on the diagonal).

Inspired by the reasoning in [YQJZ], let $2i-1$ ($1 \leq i \leq k$), be the smallest odd vertex (of rank 0) that is **unoccupied**. This means that the vertices labeled $1, 3, \dots, 2i-3$ are **occupied**. This means that the vertices with even labels, $2, \dots, 2i-2$ are **unoccupied**, and since we are talking about *order ideals*, everything $\geq$ the odd vertex $2i-1$ and above the even vertices $2, \dots, 2i-2$ gets kicked out, and for this scenario, we are left with counting order ideals of a smaller lattice, with two connected components, that consists of an even-labeled component, a triangle-lattice whose rank zero level has size n, and whose labels are $2i, 2i+2, \dots, 2i+2n-2$, and an odd-labeled component, a triangle whose rank zero level has $n-i$ vertices, and whose labels are $2i+1, 2i+3, \dots, 2n-1$. In addition we have the definitely occupied vertices



(a) The lattice $P_{13,15}$.



(b) A sub-lattice of $P_{13,15}$ which contains all order ideals of $P_{13,15}$ with smallest unoccupied odd label 5 and no consecutive labels. Note that this lattice is isomorphic to $EO(3,6)$ union the labels $1, 3$.

Figure 1

$1, \ldots, 2i-3$, but since they are definitely occupied, they don't contribute anything to the count of order ideals.

Figure 1(b) depicts the case when labels 1 and 3 of $P_{13,15}$ are occupied and 5 is empty. All vertices $\geq 5, 2, 4$ cannot be part of the order ideal.

Let $EO(a,b)$ be a two-triangle lattice, consisting of a triangle with a rank-zero vertices whose labels are $2, \ldots, 2a$, and a triangle of length-side b ($b \geq a$) whose labels are $1, 3, \ldots, 2b-1$. (See Figure 2(a) for a picture of $EO(7,9)$.) Going back to the paragraph above, subtracting $2i-1$ from all labels, gives us a lattice isomorphic to $EO(n-i,n)$. Let $e(a,b)$ be the number of order ideals of the lattice $EO(a,b)$ without consecutive labels. Then we have

$$s(n) = \sum_{i=1}^{n+1} e(n-i, n). \tag{1}$$

So if we would have an efficient 'scheme' to compute $e(a,b)$, then we would be able to compute our sequence-of-desire $s(n)$.

For $a \leq b$, let $OE(a,b)$ be $EO(b,a)$, and let $o(a,b)$ be the number of order ideals without consecutive labels of $OE(a,b)$.

By looking at the smallest unoccupied odd-labeled vertex, say $2i-1$ (see Figure 2(b)) we get, for $a \geq 1$:

$$e(a,b) = \sum_{i=1}^{b+1} o(a+1-i, b-i)\, p(i-2), \tag{2}$$

and for $a \leq 0$, we have $e(a,b) = p(b)$. Similarly, for $a \geq 1$,

$$o(a,b) = \sum_{i=1}^{a+1} e(a-i, b+1-i)\, p(i-2), \tag{3}$$

and for $a \leq 0$, we have $o(a,b) = p(b)$.

The scheme consisting of equations $(0-3)$ enables a very fast computation of the sequence $s(i)$, for, say $i \leq 400$, confirming, empirically for now, that $s(i) = 4^i$. However this can be easily turned into a fully rigorous proof. A *holonomic description* (see [Ze1], beautifully implemented by Christoph

(a) The lattice $EO(7, 9)$.

(b) A sub-lattice of $EO(7, 9)$ which contains all order ideals of $EO(7, 9)$ with smallest unoccupied odd label 9 and no consecutive labels. Note that this lattice is isomorphic to $OE(3, 4)$ union the triangular component containing the labels $1, 3, 5, 7$.

Figure 2

Koutschan in [K]) of both $e(a,b)$ and $o(a,b)$ can be readily guessed, and then, along with $p(k) = C_{k+1}$, the resulting identities $(1)-(3)$ are routinely verifiable identities in the holonomic ansatz, that can be plugged into Koutschan's 'holonomic calculator'. But since we know *a priori* that $s(k)$ satisfies *some* such recurrence, and it is extremely unlikely that its order is very high, confirming it for the first 400 values consists a convincing *semi-rigorous proof*, that is easily regorizable (if [stupidly!] desired).

3.2 Weight enumerators

But our main goal is to have $(2n+1, 2n+3)$-analogs of the work in the article [Za] that dealt with $(n, n+1)$-core partitions with distinct parts. In order to get data for the expectation, variance, and moments, we need an efficient way to generate as many terms of the sequence of *Straub polynomials*, $S_n(q)$, defined by

$$S_n(q) := \sum_p q^{size(p)},$$

where the sum ranges over all $(2n+1, 2n+3)$-core partitions with distinct parts, p, and $size(p)$ is the sum of the entries of p (i.e. the number of boxes in its Young Diagram).

The Maple package `core.txt` that accompanied [Za], and is also accompanying this article, uses Dyck paths, and was able to find the first nine Straub polynomials, $S_n(q)$, $1 \leq n \leq 9$. It is based on an extension of the method described in [EZ], but keeping track of the fact that cells with adjacent labels are not allowed. So one has to put up with much more general families of paths, that are also parametrized by a set of 'forbidden labels'. This causes an exponential expansion of memory and time.

The approach that we take in this article, that easily produced the first 21 Straub polynomials, is a weighted analog of the above naive-enumeration scheme, and goes via order ideals.

For an order ideal of $P_{m,n}$ let its *weight* be

$$q^{SumOfLabels} t^{NumberOfVertices}.$$

Let $Q(n)$ be the set of order ideals of $P_{2n+1,2n+3}$ without neighboring labels (i.e. if $a \in I$ then both $a-1$ and $a+1$ are not in I). Let's define the two-variable polynomials

$$A_n(q,t) := \sum_{I \in Q(n)} q^{SumOfLabels(I)} t^{NumberOfVertices(I)}.$$

Define the 'umbra' (linear functional on polynomials of t) by

$$U(t^k) := q^{-k(k-1)/2},$$

and extended linearly. As shown by Anderson, once $A_n(q,t)$ are known, we get $S_n(q)$ by

$$S_n(q) = U(A_n(q,t)),$$

in other words, to get $S_n(q)$ replace any power, t^k, that appears in $A_n(q,t)$, by $q^{-k(k-1)/2}$.

It remains to find an efficient scheme for 'cranking out' as many terms of $A_n(q,t)$ that our computer would be willing to compute.

We first need a weighted analog of Equation (0), i.e. the weight-enumerator of $P_{k+1,k+2}$, but we need the extra generality where (still with the smallest label being 1), for *any* positive integers c and h, in the vertical direction it is going down by c, and in the horizontal direction it going down by $c+h$ (drawing the lattice so that the highest label, $1+(c+h)(k-1)$ is at the origin, and the vertex labeled 1 is situated at the point $(k-1,0)$, and the vertex labeled $1+(k-1)h$ is situated at the point $(0,k-1)$. Note that the original $P_{k+1,k+2}$ corresponds to $c=k+1$ and $h=1$.

Let's call this generalized weight-enumerator $P_k^{(c,h)}(q,t)$. It is readily seen that the weighted analog of Eq. (0) is

$$P_k^{(c,h)}(q,t) \; = \; \sum_{i=1}^{k+1} t^{i-1} \, \cdot \, q^{(i-1)+(i-1)(i-2)h/2} \, \cdot \, P_{i-2}^{(c,h)}(q, q^{c+h}t) \, \cdot \, P_{k-i}^{(c,h)}(q, q^{ih}t), \tag{0w}$$

with the initial conditions $P_{-1}=1, P_0=1$.

Let $E_{x,y}^{(c)}(q,t)$ be the weight-enumerator of the lattice $EO(x,y)$ with horizontal spacing c and vertical spacing $c+2$. Then the analog of Eq. (1) is

$$A_n(q,t) \; = \; \sum_{i=1}^{n+1} t^{i-1} q^{(i-1)^2} \, \cdot \, E_{n-i,n}^{(2n+1)}(q, q^{2i-1}t). \tag{1w}$$

Let $O^{(c)}_{x,y}(q,t)$ be the weight-enumerator of the lattice $OE(x,y)$, with horizontal spacing c and vertical spacing $c+2$. Then the analog of Eq. (2) can be seen to be

$$E^{(c)}_{x,y}(q,t) \;=\; \sum_{i=1}^{y+1} t^{i-1} \cdot q^{(i-1)^2} \cdot O^{(c)}_{x-i+1,y-i}(q,q^{2i-1}t) \cdot P^{(c,2)}_{i-2}(q,q^{c+2}t), \quad (2w)$$

with the *initial condition* $E^{(c)}_{x,y}(q,t) = P^{(c,2)}_{y}(q,t)$ when $x \le 0$.

Finally, the weighted analog of Eq. (3) is

$$O^{(c)}_{x,y}(q,t) \;=\; \sum_{i=1}^{x+1} t^{i-1}\, q^{(i-1)^2} \cdot E^{(c)}_{x-i,y-i+1}(q,q^{2i-1}t) \cdot P^{(c,2)}_{i-2}(q,q^{c+2}t), \quad (3w)$$

with the *initial condition* $O^{(c)}_{x,y}(q,t) = P^{(c,2)}_{y}(q,qt)$ when $x \le 0$.

3.3 The first 21 Straub polynomials

Using the above scheme, one gets that

$$S_1(q) \;=\; q^4+q^2+q+1,$$

$$S_2(q) \;=\; q^{21}+q^{16}+2\,q^{12}+q^9+q^8+q^7+q^6+q^5+2\,q^4+2\,q^3+q^2+q+1$$

$$S_3(q) \;=\; q^{65}+q^{56}+q^{48}+q^{47}+q^{41}+q^{39}+q^{37}+2\,q^{35}+q^{32}+q^{30}+2\,q^{29}+q^{28}+q^{26}+3\,q^{24}+q^{23}+q^{22}$$
$$+q^{21}+q^{20}+2\,q^{19}+2\,q^{18}+3\,q^{17}+q^{16}+q^{15}+2\,q^{14}+2\,q^{13}+2\,q^{12}+3\,q^{11}+q^{10}+3\,q^9+3\,q^8$$
$$+3\,q^7+4\,q^6+3\,q^5+2\,q^4+2\,q^3+q^2+q+1,$$

$$S_4(q) \;=\; q^{155}+q^{141}+q^{128}+q^{125}+q^{116}+q^{112}+2\,q^{105}+q^{103}+q^{100}+2\,q^{95}+q^{93}+q^{91}+2\,q^{89}+q^{85}+q^{84}$$
$$+q^{83}+2\,q^{82}+q^{80}+q^{79}+q^{78}+q^{76}+q^{74}+q^{73}+q^{72}+2\,q^{71}+2\,q^{70}+q^{69}+2\,q^{68}+q^{67}+q^{65}+q^{64}$$
$$+q^{63}+5\,q^{61}+q^{60}+2\,q^{59}+3\,q^{57}+q^{56}+3\,q^{55}+4\,q^{53}+2\,q^{52}+2\,q^{51}+2\,q^{50}+q^{49}+2\,q^{48}+3\,q^{47}$$
$$+2\,q^{46}+3\,q^{45}+4\,q^{44}+2\,q^{43}+q^{42}+5\,q^{40}+3\,q^{39}+4\,q^{38}+5\,q^{37}+2\,q^{36}+3\,q^{35}+q^{34}+4\,q^{33}$$
$$+6\,q^{32}+5\,q^{31}+3\,q^{30}+4\,q^{29}+3\,q^{28}+5\,q^{27}+4\,q^{26}+7\,q^{25}+5\,q^{24}+6\,q^{23}+3\,q^{22}+4\,q^{21}+5\,q^{20}$$
$$+5\,q^{19}+4\,q^{18}+5\,q^{17}+6\,q^{16}+5\,q^{15}+4\,q^{14}+7\,q^{13}+6\,q^{12}+7\,q^{11}+7\,q^{10}+6\,q^9+6\,q^8+5\,q^7$$
$$+4\,q^6+3\,q^5+2\,q^4+2\,q^3+q^2+q+1.$$

For the Straub polynomials $S_n(q)$ for $5 \leq n \leq 21$, see the webpage `http://www.math.rutgers.edu/~zeilberg/tokhniot/oArmin3.txt` , or use procedure `ASpc(n,q)` in the Maple package `Armin.txt` mentioned above.

Unlike the case of $(s, s+1)$-core partitions, whose number happened to be F_{s+1}, and the explicit expressions for the expectation, variance, and higher moments involved expressions in F_s, F_{s+1} and s, the present case of $(2n+1, 2n+3)$-core partitions into distinct parts, gives, surprisingly, 'nicer' results. This is because, as conjectured in [Str] and first proved in [YQLZ] (and reproved above), the actual enumeration is as simple as can be, namely 4^n. Hence it is not surprising that the expectation, variance, and higher moments are *polynomials* in n.

To get expressions for the moments we used the empirical-yet-rigorizable approach of [Ze2] and [Ze3], as follows.

Using the first 21 Straub polynomials, we get the sequence of numerical averages $S'_n(1)/4^n$, $1 \leq n \leq 21$, and 'fit it' to a polynomial of degree 3 (in fact four terms suffice!), we get the expression for the expectation, let's call it $\mu(n)$, stated in Theorem 1 above.

Using the sequence

$$\frac{(s\frac{d}{ds})^2 S_n(q)|_{q=1}}{4^n} - \mu(n)^2,$$

for $1 \leq n \leq 7$, and 'fitting' it with a polynomial of degree 6, we get an explicit expression for the variance, thereby getting Theorem 2. The conjectured polynomial expression agrees all the way to $n = 21$.

The third-through the seventh moments are derived similarly, where the i-th moment (about the mean, but also the straight moment) turns out to be a polynomial of degree $3i$ in n.

Let us comment that all the results here can be, *a posteriori*, justified rigorously. The complicated functional recurrences for the Straub polynomials (before the "umbral application") entail, after Taylor expansions about $q = 1$, extremely complicated recurrence relations for the (pre-) moments, whose details do not concern us, since we know that their truth follows by induction. Each such identity is a *polynomial identity*, and hence its truth follows from plugging-in sufficiently many special cases. But that's how we got them in the first place. **QED**!

3.4 Encore: A one-line proof of Straub's ex-conjecture about the maximal size of a $(2n+1, 2n+3)$ core partition into distinct parts

In [YQLZ], the authors used quite a bit of *human ingenuity* to prove Armin Straub's conjecture (posed in [Str]) that the maximal size of a $(2n+1, 2n+3)$-core partition into distinct parts is given by the degree-4 polynomial $\frac{1}{24}\,(5\,n+11)\,n\,(n+2)\,(n+1)$.

But since it is clear, from general, *a priori*, *hand-waving* (yet fully rigorous) considerations that this quantity is *some* polynomial of degree ≤ 5, it is enough to check it for $1 \leq n \leq 6$. But this quantity is exactly the **degree** of the Straub polynomial $S_n(q)$. We verified it, in fact, all the way to $n = 21$, so Theorem $0'$ is re-proved (with a *vengeance*!).

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