Using Symbolic Computation to Explore Generalized Dyck Paths and Their Areas

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Dedicated to Symbolic Computation pioneer Bruno Buchberger (b. 22 October, 1942) on his 80^{th} *birthday*

Abstract: We show the power of Bruno Buchberger's seminal Gröbner Basis algorithm, interfaced, seamlessly, with what we call *symbolic dynamical programming*, to *automatically* generate algebraic equations satisfied by the generating functions enumerating so-called *Generalized Dyck Walks*, i.e. 2D walks that start and end on the x-axis, and never dip below it, for an *arbitrary* set of steps. More impressively, we combine it with calculus (that Maple knows very well!), to automatically compute generating functions for the sum-of-the-areas of these generalized Dyck paths, and even for the sum of any given power of the areas, enabling us to get statistical information about the area under a random generalized Dyck path.

Maple package and Output Files

This article is accompanied by a Maple package, `GDW.txt`, and numerous output files, some of which will be referred to later.

They are all obtainable from the front of this article

`https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/area.html` .

Bruno Buchberger: the Gauss of Non-Linear Algebra

Using *Gaussian elimination*, we can solve, efficiently, any system of **linear equations** with any (finite) number of unknowns. Using the **Buchberger algorithm** [B1] [B2] (see the modern classic [CLO] for a lucid and engaging account) we can solve any system of **non-linear equations** with any (finite) number of unknowns. Alas, due to the non-linearity, we can only go so far, but yet, with modern computers, and *computer algebra systems* (our favorite being Maple), one can do a lot.

In the applications that we need, the scenario is that we have N quantities $X_1, \ldots, X_N$ and they satisfy N polynomial equations

$$P_i(t; X_1, \ldots, X_n) = 0 \quad , \quad 1 \leq i \leq N \quad ,$$

where t is variable (or multi-variable), that may be viewed as a parameter. We are **really** only interested in the first quantity, $X_1 = X_1(t)$.

In our application, these equations are derived using combinatorial considerations, that we can teach to the computer. There is our **primary** object, X_1, that we really care about, but in order to determine it, we need to introduce **secondary** objects, $X_2, \dots, X_n$, that we don't care about, and whose sole *raison d'être* is that they enable us to get a grip on X_1.

All we want is a **succinct** algebraic relation of the form $Q(X_1, t) = 0$, where Q is a bi-variate polynomial of X_1 and t.

In order to **eliminate** the unwanted quantities $X_2, \dots, X_n$, and get a pure equation only involving X_1 (and the auxiliary variable (or variables), t), the command in `Maple` is:

$$Groebner[Basis](\{P_1, P_2, \dots, P_n\}, plex(X_2, X_3, \dots, X_n, X_1))[1]; \, .$$

To take a random example (we just made up), suppose that we have the system of three equations in the three unknowns X, Y, Z that depend on some auxiliary variable t:

$$X = Y^2 + Z^2 + 1 \quad , \quad Y = X^2 + 3Z^2 + t \quad , \quad Z = XYZ + t + 1 \quad ,$$

and we want to **eliminate** Y and Z, i.e. get an algebraic equation in $X = X(t)$ that only involves, in addition to X, the variable t. Then type in a Maple session

```
Groebner[Basis]( { X-Y**2-Z**2-1,Y-X**2-3*Z**2-t, Z-X*Y*Z-t-1 } ,plex(Y,Z,X))[1]=0;
```

getting that our quantity of interest $X = X(t)$, satisfies the following degree-12 polynomial equation in $X(t)$ (abbreviated X), with coefficients that are polynomials of t:

$$X^{12} + 6X^{11} + (4t+3)\,X^{10} + (18t-21)\,X^9 + \left(6t^2-8\right)X^8 + \left(18t^2-44t-9\right)X^7$$
$$+ \left(4t^3+9t^2+20t+60\right)X^6 + \left(6t^3+29t^2+96t+43\right)X^5 + \left(t^4+30t^3+16t^2-14t-24\right)X^4$$
$$+ \left(52t^3+177t^2+232t+50\right)X^3 + \left(18t^4-12t^3+89t^2+242t+149\right)X^2$$
$$+ \left(54t^3+12t^2-150t-105\right)X + 81t^4 + 378t^3 + 612t^2 + 396t + 99 \; = \; 0 \quad .$$

The Art of Counting in General

In *enumerative combinatorics*, we are interested in *counting* families of sets $\{A(n)\}$, defined by some combinatorial conditions, indexed by (one or more) discrete parameter(s) n. The best possible scenario is an **explicit** (aka *closed form*) expression for $a(n) := |A(n)|$, where as usual $|S|$ denotes the number of elements of the finite set S.

For example, the number of words in the alphabet $\{0, 1\}$ of length n (Ans.: 2^n), or the number of permutations of length n (Ans.: $n!$).

Alas, in many cases there is no *closed form* (unless you consider $\sum_{s \in A(n)} 1$ as one), and one looks for a closed-form, or failing this, an *equation*, satisfied by the (ordinary) generating function

$$X(t) \; = \; \sum_{n=0}^{\infty} a(n) t^n \quad .$$

In many applications, including all those in the present paper, it is better to think of $X(t)$ as a *weight enumerator* of the (usually) 'infinite' set

$$\mathcal{A} := \bigcup_{n=0}^{\infty} A(n) \quad ,$$

and define, $Weight(s) := t^n$ if $s \in A(n)$. The generating function, $X(t)$, is the **total weight** (i.e. sum of all the weights of its members) of $\mathcal{A}$, denoted by $Weight(\mathcal{A})$.

Reminders about Counting Classical Dyck and Motzkin Walks

In order to illustrate our general approach, already initiated in [AyZ], and further pursued in [TZ], [EK1], and [EK2], let's revisit the very classical problem of counting (ungeneralized) **Dyck walks**. These are walks that start at the origin $(0,0)$, end somewhere on the x-axis, using the **atomic steps** $[1,1], [1,-1]$ (i.e. moving from (x,y) to either $(x+1,y+1)$ or $(x+1,y-1)$, respectively) and **crucially**, always stay in the upper-half plane $y \geq 0$.

The weight of a walk w is $t^{NumberOfSteps(w)}$, in other words t^n, if the endpoint of our walk is $(n,0)$.

Let $f[0,0](t)$ be our object of desire, the weight-enumerator of all Dyck walks. There are two possibilities.

• The walk is empty (of length 0), with weight $t^0 = 1$.

• The walk, let's call it w, sooner of later, again touches the x-axis. Let w_1 be the prefix of w consisting of this portion and let w_2 be the rest, so we can express w as the **catenation** $w = w_1\, w_2$, where w_2 is a (possibly empty) shorter walk weight-counted by $f[0,0](t)$, but w_1 is a **strict Dyck path**, a walk that starts and ends on the x-axis, and except at the endpoints, is **strictly** above the x-axis. Let's call the weight-enumerator of these kind of walks $g[0,0](t)$.

Taking weights, we get a first equation

$$f[0,0](t) = 1 + g[0,0](t) \cdot f[0,0](t) \quad .$$

Note that we were **forced** to introduce the quantity $g[0,0](t)$.

Let's look at the anatomy of such a walk that is weight-enumerated by $g[0,0](t)$. Since it is non-empty, and can't get below the x-axis, it **must** start with the step $[1,1]$, i.e. the first step is from the origin $(0,0)$ to $(1,1)$. Since sooner or later it must return to the x-axis, its last step must be a down step $[1,-1]$, from $(n-1,1)$ to $(n,0)$ (say). Removing this first and last step is a walk that is shorter by two steps that is *weakly* above the line $y=1$. But these walks are in obvious bijection with walks weight-counted by $f[0,0](t)$. Hence we get a **second** equation

$$g[0,0](t) \;=\; t^2 \cdot f[0,0](t) \quad .$$

So we get a system of two equations and two unknowns.

In this case, we don't need a computer, or Bruno Buchberger, to solve this system, and eliminating $g[0,0](t)$ gives the quadratic equation for $f[0,0](t)$ (let's abbreviate it to $f[0,0]$)

$$t^2 f[0,0]^2 - f[0,0] + 1 = 0 \quad ,$$

that thanks to the Babylonians can be solved in terms of radicals, yielding the Catalan numbers.

Similarly, for **Motzkin walks**, where the set of steps is $\{[1,1],[1,0],[1,-1]\}$, we still only have one extra quantity $g[0,0](t)$, and the two equations are

$$\{f[0,0] = 1 + tf[0,0] + g[0,0] \cdot f[0,0] \quad , \quad g[0,0] = t^2 \cdot f[0,0]\} \quad ,$$

that again lead to a simple quadratic equation, whose solution is the generating function for Motzkin numbers.

The primary novelty of the present article is to keep track of the **area** statistics. Let $f[0,0](t,q)$, that we will still call $f[0,0](t)$ for short (not to be confused with the previous $f[0,0](t)$), but it is implied that there is also a variable q, that may be viewed as a parameter, be the weight-enumerator of Dyck walks according to the bi-variate weight

$$Weight(w) := t^{LengthOf(w)}\, q^{AreaUnder(w)} \quad .$$

One way to define the 'area' without geometry is as the sum of the y-coordinates of all the intermediate points on the walk.

Going back to the classical Dyck case, instead of a system of *algebraic equations*, we get a system of *functional equations*. It is easy to see that we have .

$$\{f[0,0](t) = 1 + g[0,0](t) \cdot f[0,0](t) \quad , \quad g[0,0](t) = qt^2 \cdot f[0,0](qt)\} \quad .$$

Of course one can eliminate $g[0,0](t)$ and get a pure functional equation for $f[0,0](t)$ (let's call it $X(t)$).

$$X(t) = 1 + qt^2 \cdot X(t) \cdot X(qt) \quad .$$

Alas $f[0,0](t) = X(t)$ is no longer an algebraic formal power series.

The study of the area under Dyck paths was pioneered in [MSV] and further explored in [Wo],[SRW], and for higher moments by Robin Chapman [C]. We will soon see how to fully automate this to a much more general setting. In particular, our Maple package, `GDW.txt`, can confirm, in a few nano-seconds, all their beautiful human-generated results.

Similarly, for the Motzkin case, the system of functional equations is:

$$\{f[0,0](t) = 1 + t \cdot f[0,0](t) + g[0,0](t) \cdot f[0,0](t) \quad , \quad g[0,0](t) = qt^2 \cdot f[0,0](qt)\} \quad ,$$

and the pure functional equation for $f[0,0](t)$ (again let's call it $X(t)$) is:

$$X(t) = 1 + t \cdot X(t) + qt^2 \cdot X(t) \cdot X(qt) \quad .$$

The bi-variate generating function contains lots of statistical information. To get the average area under a random Dyck walk, we must first compute the generating function for the 'sum of the areas of all walks' that is given by

$$\frac{d}{dq} f[0,0](t)\Big|_{q=1} \quad ,$$

(recall that $f[0,0](t)$ is really a function of both t and q.). More generally, the generating function for the sum of the r-th powers of the areas, is

$$\left(q\frac{d}{dq}\right)^r f[0,0](t)\Big|_{q=1} \quad .$$

If you are interested in these quantities up to the r-th power of the area, we can taylor expand $f[0,0](t)$ (and similarly $g[0,0](t)$) as a Taylor expansion about $q=1$:

$$f[0,0](t) = f[0,0,0](t) + f[0,0,1](t)(q-1) + f[0,0,2](t)(q-1)^2 + \ldots + f[0,0,r](t)(q-1)^r + \ldots$$

where $f[0,0,0](t), f[0,0,1](t), \ldots$ are formal power series of t alone, and do not depend on q. (Note that what we now call $f[0,0,0](t)$ is identical to what we called $f[0,0](t)$ in the original, straight-enumeration case).

It would follow from the algorithm to be described shortly, that these are all **algebraic** formal power series, and thanks to the **Buchberger Gröbner basis algorithm**, we can actually find the *pure* equations that they each satisfy.

Plugging this *series expansion* into the functional equation, would introduce quantities like $f[0,0,0](qt)$.

For this we need an elementary lemma from Calculus:

Simple Lemma: Let $f(t)$ be a formal power series of a single variable t, and q be another variable, then

$$f(qt) \;=\; f(t) + (q-1)tf'(t) + \frac{1}{2}(q-1)^2t^2f''(t) + \ldots + \frac{1}{r!}(q-1)^r t^r f^{(r)}(t) + \ldots \quad .$$

So we can Taylor-expand it with respect to q around $q=1$, and thanks to Calculus, that Maple knows so well, *automatically* express it in terms of $f[0,0,0]'(t), f[0,0,1]'(t)\ldots$. It seems that we have too many quantities, and that we need more equations. But we can also (automatically!) differentiate with respect to t the algebraic equation satisfied by $f[0,0,0](t)$, (and if needed, by $f[0,0,1](t)$ etc.) using *implicit differentiation* (that Maple also knows!), and collect terms, and at the **end of the day** we would have as many algebraic equations as quantities. If we want to focus, say, on $f[0,0,1](t)$ (alias $(\frac{\partial}{\partial q})f[0,0](t)\Big|_{q=1}$, alias the generating function for the 'sum of areas' of the walks), we can use Buchberger's algorithm as above with *plex*, singling out $f[0,0,1](t)$. Ditto for higher (factorial) moments, i.e. $f[0,0,2](t), f[0,0,3](t), \ldots$ except that things do get more and more complicated, even for computerkind.

Straight Enumeration of Generalized Dyck Paths

Let's recall the method initiated in [AyZ], extended in [EK1] and [EK2], and continued in [TZ], but with a new implementation, that makes full use of Gröbner bases. Alternative approaches to the problem of (straight) enumeration have been undertaken by Banderier et. al. [BKKKKNW], using the *kernel method*, and in [EkhZ], using *"Guess and Check"*.

The **input** is an arbitrary set of integers, S, and our goal is to find a pure algebraic equation of the form

$$\sum_{i=0}^{d} p_i(t)X(t)^i \quad = 0,$$

where $p_i(t)$ are polynomials in t for the following formal power series

$$X(t) = \sum_{n=0}^{\infty} a_S(n)\, t^n \quad ,$$

where $a_S(n)$ is the number of walks from $(0,0)$ to $(n,0)$, in the 2D lattice, with the set of allowable steps

$$(x,y) \rightarrow (x+1, y+s) \quad , \quad s \in S \quad ,$$

and that always stay in the upper half plane $y \geq 0$. Equivalently, and more useful for us, it is the *weight enumerator*, according to the weight $t^{NumberOfSteps}$, of the (infinite) set of all these walks.

Remark: If you don't insist on always staying above the x-axis, then the number of such n-step walks from the origin to the x-axis, is the constant term of

$$\left(\sum_{s \in S} t^s\right)^n \qquad ,$$

and it is possible, very fast, to use the **Almkvist-Zeilberger algorithm** [AlZ] (see [D] for a great exposition) to get a linear recurrence equation with polynomial coefficients.

Note that if S only consists of positive integers, or only consists of negative integers, then $X(t)$ is trivially 1. So for things to be non-trivial, S must have at least one positive member and at least one negative member.

We will rename $X(t)$, $f[0,0](t)$, since we need to introduce auxiliary quantities $f[a,b](t)$ and $g[a,b](t)$.

For integer $a \geq 0$ and $b \geq 0$,

• Let $f[a,b](t)$ be the weight-enumerator of walks with a set of steps given by S, that start at the point $(0,a)$ and end on the horizontal line $y=b$ and stay *weakly* above the x-axis.

• Let $g[a,b](t)$ be the weight-enumerator of non-empty walks with a set of steps given by S, that start at the point $(0,a)$ and end on the horizontal line $y=b$ and stay *strictly* above the x-axis, except at an endpoint when $a = 0$ or $b = 0$. If both $a > 0$ and $b > 0$ then we have no need for it, and we declare that it is 0.

For each $f[a, b](t)$ and $g[a, b](t)$ that will show up we would need to set its own equation.

How to form the Equations for the $f[a, b]$?

• If $a > 0$ and $b > 0$ then

$$f[a, b] = g[a, 0] \cdot f[0, b] + f[a - 1, b - 1] \quad .$$

Explanation: if such a walk, that starts at $y = a$ and ends at $y = b$, touches the x-axis, then the first portion until that first encounter is a walk weight-counted by $g[a, 0]$, while the second part is a walk weight-counted by $f[0, b]$ (it starts at $y = 0$, ends at $y = b$ and stays weakly above the x-axis). On the other hand, if it never touches the x-axis it must stay in $y \geq 1$, so lowering it by 1 unit, yields a walk weight-counted by $f[a - 1, b - 1]$.

• If $a > 0$ and $b = 0$ then

$$f[a, 0] = g[a, 0] \cdot f[0, 0] \quad .$$

Explanation: A walk that starts at $(0, a)$ and ends on $y = 0$, must meet the x-axis for the first time, this initial portion is weight-counted by $g[a, 0]$, the remaining portion of the walk is weight-counted by $f[0, 0]$.

Similarly

• If $a = 0$ and $b > 0$ then

$$f[0, b] = f[0, 0] \cdot g[0, b] \quad .$$

• If $a = 0$ and $b = 0$ and $0 \notin S$

$$f[0, 0] = 1 + g[0, 0] \cdot f[0, 0] \quad .$$

Explanation: A walk that starts and ends on $y = 0$ may be the empty walk (weight=1). If not, it can be broken up into the portion until the first encounter with the x axis, that is weight-counted by $g[0, 0]$, followed by any-old (shorter) walk weight-counted by $f[0, 0]$.

• If $a = 0$ and $b = 0$ and $0 \in S$

$$f[0, 0] = 1 + t \cdot f[0, 0] + g[0, 0] \cdot f[0, 0] \quad .$$

Explanation: Now the first step could also be $(0, 0) \to (1, 0)$, and after that it is a typical walk weight-counted by $f[0, 0]$.

This is implemented in procedure `MakeEqF(f,g,x,a,b,S)` in our Maple package. In fact, it is more general, keeping track of the individual steps.

How to form the Equations for the $g[a,b]$?

We also need to set up equations for $g[a,b]$ for those (a,b) that would be needed.

Let P be the subset of S consisting of the (strictly) positive members of S, and let N be the subset of S consisting of the (strictly) negative members of S, so if $0 \in S$ then

$$S = P \cup N \cup \{0\} \quad ,$$

while, if $0 \notin S$ then

$$S = P \cup N \quad .$$

• If $a = 0$ and $b > 0$ then

$$g[0,b] = t \left(\sum_{i \in P} f[a+i-1, b-1] \right) \quad .$$

Explanation: Such a walk must start with a step of the form $[1,i]$ where $i \in P$, and then it is always in $y \geq 1$. Removing that first step (weight t) and 'lowering' it by 1 unit, is in bijection with a walk weight-counted by $f[a+i-1, b-1]$.

• If $a > 0$ and $b = 0$ then

$$g[a,0] = t \left(\sum_{j \in N} f[a-1, b-j-1] \right) \quad .$$

Explanation: Such a walk must end with a step of the form $[1,j]$ where $j \in N$, and before that it is always in $y \geq 1$. Removing that last step (weight t) and 'lowering' it by 1 unit, is in bijection with a walk weight-counted by $f[a-1, b-j-1]$.

• If $a = 0$ and $b = 0$ then

$$g[0,0] = t^2 \left(\sum_{i \in P} \sum_{j \in N} f[a+i-1, b-j-1] \right) \quad .$$

Explanation: Every non-empty walk that starts at $y = 0$ and ends at $y = 0$ and that never touches the x-axis except at the two endpoints, must start with a positive step $[1,i]$ and end with a negative step $[1,j]$. Deleting the first and last steps gives you a shorter walk (with two steps shorter) that is always in the region $y \geq 1$. Lowering the remaining walk by 1 unit gives a walk weight-counted by $f[a+i-1, b-j-1]$.

This is implemented in procedure `MakeEqG(f,g,x,a,b,S)` in our Maple package. Again, it is more general keeping track of the individual steps.

To get the full system of equations with a set of steps `S`, type:

MakeSysT(f,g,t,S);

It returns the set of equations, followed by the set of quantities that participate. For example

`MakeSysT(f,g,t, { 1,2,-1,-2 } )[1];` gives

$$\{f_{00} = f_{00}g_{00} + 1 \quad , \quad f_{01} = f_{00}g_{01} \quad , \quad f_{10} = g_{10}f_{00} \quad , \quad f_{11} = f_{01}g_{10} + f_{00},$$
$$g_{00} = t^2 f_{00} + t^2 f_{01} + t^2 f_{10} + t^2 f_{11} \quad , \quad g_{01} = t f_{00} + t f_{10} \quad , \quad g_{10} = t f_{00} + t f_{01}\} \quad ,$$

while `MakeSysT(f,g,t, {1,2,-1,-2 })[2];` gives the set of quantities

$$\{f_{00} \quad , \quad f_{01} \quad , \quad f_{10} \quad , \quad f_{11} \quad , \quad g_{00} \quad , \quad g_{01} \quad , \quad g_{10}\} \quad .$$

Weighted Enumeration of Generalized Dyck Paths According to the Area

We can teach Maple how, *all by itself*, set up a system of **functional equations**, for the q-analog, where one keeps track of the area. Everything is analogous.

For integer $a \geq 0$ and $b \geq 0$,

• Let $f[a,b](t)$ be the weight-enumerator of walks with a set of steps given by S, that start at the point $(0,a)$ and end on the horizontal line $y = b$ and stay *weakly* above the x-axis, but now the *weight* of a walk is not just t^{length} but rather $t^{length} \cdot q^{AreaUnder}$. For the sake of notational convenience, We still write $f[a,b](t)$ rather than $f[a,b](t,q)$, but one should keep in mind that they all depend on q. We can think of q as a parameter.

Similarly,

• Let $g[a,b](t)$ be the weight-enumerator (in the above sense of $t^{length} \cdot q^{AreaUnder}$) of non-empty walks with a set of steps given by S, that start at the point $(0,a)$ and end on the horizontal line $y = b$ and stay *strictly* above the x-axis, except at an endpoint when $a = 0$ or $b = 0$. If both $a > 0$ and $b > 0$ then we have no need for it and we declare that it is 0.

How to form the Functional Equations for $f[a,b](t)$?

• If $a > 0$ and $b > 0$ then

$$f[a,b](t) = g[a,0](t) \cdot f[0,b](t) \ + \ f[a-1,b-1](q\,t) \quad .$$

Explanation: if such a walk that starts at $y = a$ and ends at $y = b$ touches the x-axis, then the first portion until that first encounter is a walk weight-counted by $g[a,0](t)$, while the second part is a walk weight-counted by $f[0,b](t)$ (it starts at $y = 0$, ends at $y = b$ and stays weakly above the x-axis). On the other hand, if it never touches the x-axis, it must stay in $y \geq 1$, so lowering it by 1 unit, results in a walk weight-counted by $f[a-1,b-1](t)$. **But**, by lowering it, we lost some area!. If the length of the walk is n, then we lost n units of area (the area of an $n \times 1$ rectangle), hence this is bi-weight-enumerated by $f[a-1,b-1](q\,t)$, (rather than $f[a-1,b-1](t)$.)

• If $a>0$ and $b=0$ then

$$f[a,0](t) \,=\, g[a,0](t) \cdot f[0,0](t) \quad .$$

Explanation: Each walk that starts at $y=a$ must meet the x-axis for the first time, this portion is weight-counted by $g[a,0](t)$, the remaining portion of the walk is weight-counted by $f[0,0](t)$.

Similarly:

• If $a=0$ and $b>0$ then

$$f[0,b](t) \,=\, f[0,0](t) \cdot g[0,b](t) \quad ,$$

• If $a=0$ and $b=0$ and $0 \notin S$

$$f[0,0](t) \,=\, 1 \,+\, g[0,0](t) \cdot f[0,0](t) \quad .$$

Explanation: Each walk that starts and ends on $y=0$ must either be the empty walk (of length 0, area 0, and hence with weight $t^0 \cdot q^0 = 1$), or else meet the x-axis for the first time (weight-counted by $g[0,0](t)$), followed by any-old walk (possibly empty) weight-counted by $f[0,0](t)$.

• If $a=0$ and $b=0$ and $0 \in S$

$$f[0,0](t) \,=\, 1 \,+\, t \cdot f[0,0](t) \,+\, g[0,0](t) \cdot f[0,0](t) \quad .$$

Explanation: Now the first step could also be $(0,0) \rightarrow (1,0)$, and after that it is a typical walk weight-counted by $f[0,0](t)$, and there is no area gain.

This is implemented in procedure `qMakeEqF(f,g,t,q,a,b,S)` in our Maple package.

How to form the Functional Equations for $g[a,b](t)$?

We also need to set up equations for $g[a,b](t)$ for those (a,b) that would be required.

As before:

Let P be the subset of S consisting of the (strictly) positive members of S, and let N be the subset of S consisting of the (strictly) negative members of S, so if $0 \in S$ then

$$S = P \cup N \cup \{0\} \quad ,$$

while, if $0 \notin S$ then

$$S = P \cup N \quad .$$

• If $a = 0$ and $b > 0$ then

$$g[0,b](t) = t\left(\sum_{i \in P} q^{i/2}\, f[a+i-1,b-1](qt)\right) \quad .$$

Explanation: Such a walk must start with a step of the form $[1,i]$ where $i \in P$, and then it is always in $y \geq 1$. Removing that first step reduces the area by $1 \times i/2 = i/2$ and 'lowering' by 1 unit, it looses n units of area (if the remaining path has length n), to account for the lost area we need $f[a+i-1,b-1](qt)$, (rather than $f[a+i-1,b-1](t)$).

Similarly:

• If $a > 0$ and $b = 0$ then

$$g[a,0](t) = t\left(\sum_{j \in N} q^{-j/2} f[a-1,b-j-1](qt)\right) \quad .$$

• If $a = 0$ and $b = 0$ then

$$g[0,0](t) = t^2\left(\sum_{i \in P}\sum_{j \in N} q^{i/2-j/2}\, f[a+i-1,b-j-1](qt)\right) \quad .$$

Explanation: Every walk that starts at $y = 0$ and ends at $y = 0$ and that never touches the x-axis except at the endpoints, must start with a positive step $[1,i]$ and end with a negative step $[1,j]$. Deleting the first and last steps gives you a shorter walk (with two steps shorter) that is always in the region $y \geq 1$. Removing the first step (i.e. $(0,0) \to (1,i)$) reduces the area by $i/2$. Removing the last step (i.e. $(n-1,-j) \to (n,0)$) reduces the area by $-j/2$.

Lowering it by 1 unit gives a walk weight-counted by $f[a+i-1,b-j-1](qt)$.

This is implemented in procedure `qMakeEqGt(f,g,t,q,a,b,S)` in our Maple package.

To get the full system of functional equations, followed by the quantities that feature in them, with set of steps `S`, type:

`qMakeSysT(f,g,t,q,S);` .

For example, if the set of steps is $S = \{2,1,0,-1,-2\}$, typing

`qMakeSyst(f,g,t,q, { 2,1,0,-1,-2 } )[1];`

gives

$$\{f_{01}(t) - f_{00}(t)\, g_{01}(t) \quad , \quad f_{10}(t) - g_{10}(t)\, f_{00}(t) \quad ,$$

$$f_{11}(t) - f_{01}(t)\, g_{10}(t) - f_{00}(qt) \quad , \quad g_{01}(t) - t\sqrt{q}\, f_{00}(qt) - tq f_{10}(qt) \quad ,$$

$$g_{10}(t) - tq f_{01}(qt) - t\sqrt{q}\, f_{00}(qt)\,, \quad f_{00}(t) - f_{00}(t)\, g_{00}(t) - f_{00}(t)\, t - 1 \quad ,$$

$$g_{00}(t) - t^2 q^{\frac{3}{2}} f_{01}(qt) - t^2 q f_{00}(qt) - t^2 q^2 f_{11}(qt) - t^2 q^{\frac{3}{2}} f_{10}(qt)\} \quad ,$$

while to see the set of featured quantities type

`qMakeSyst(f,g,t,q, { 2,1,0,-1,-2 })[2];` , getting

$$\{f_{00}(t) \quad , \quad f_{00}(qt) \quad , \quad f_{01}(t) \quad , \quad f_{01}(qt) \quad , \quad f_{10}(t) \quad , \quad f_{10}(qt) \quad ,$$

$$f_{11}(t) \quad , \quad f_{11}(qt) \quad , \quad g_{00}(t) \quad , \quad g_{01}(t) \quad , \quad g_{10}(t)\} \quad .$$

From Functional Equations to Algebraic Equations

After the computer finds the system of *functional* equations described above, we instruct it to find a system *algebraic* equations for the 'components' of the $f[a,b](t)$ (and we also need $g[a,b](t)$). If we are interested in finding the pure algebraic equation satisfied by the formal power series $f[0,0,0](t), f[0,0,1](t), \dots, f[0,0,k](t)$. We write

$$f[0,0](t) = \sum_{i=0}^{k} f[0,0,i](t)(q-i)^i + O((q-1)^{k+1}) \quad ,$$

and analogously for the other $f[a,b](t)$ and $g[a,b](t)$, then expand in powers of $(q-1)$, collect terms, use the Lemma, and get more equations by differentiating with respect to t each of these equations up to the k-th derivative, using implicit differentiation.

Because of the extreme complexity, we decided only to implement this scheme for $k=1$, i.e. for finding the algebraic equation satisfied by the generating function for the 'sum of the areas'.

This is implemented in procedure `qEqGFt(S,X,t)` . For example, to get the algebraic equation for the generating function for 'sum of areas' of the classical Dyck paths, type:

```
qEqGFt({1,-1},X,t);
```

getting

$$t^2 \,-\, (4t^2-1)(2t^2-1)X \,+\, t^2(4t^2-1)^2\, X^2 \,=\, 0 \quad .$$

(This is A8549 of [Sl], `https://oeis.org/A008549`).

For Motzkin walks, typing

`qEqGFt({1,0,-1},X,t);`, gives

$$t^2 - (3t-1)\,(t+1)\left(t^2+2t-1\right) X + t^2\,(3t-1)^2\,(t+1)^2\, X^2 \,=\, 0 \quad .$$

(This is A57585 of [Sl], `https://oeis.org/A057585`).

For a more complicated example, to get the pure algebraic equation satisfied by the generating function for the 'sum of the areas under generalized Dyck paths with set of steps $\{[1,2],[1,1],[1,0],[1,-1],[1,-2]\}$', type:

`qEqGFt({ 2,1,0,-1,-2 },X,t);`, getting, after less than a minute,

$$t^2\left(775t^4-1460t^3+1006t^2-264t+24\right)$$
$$+\left(t-1\right)\left(5t-1\right)\left(425t^6-1520t^5+1527t^4-68t^3-282t^2+88t-8\right)X$$
$$-t\left(150t^5+540t^4-889t^3-240t^2+228t-32\right)\left(t-1\right)^2\left(5t-1\right)^2X^2$$
$$-2t^2\left(5t+4\right)\left(5t^3-t^2-17t+4\right)\left(t-1\right)^3\left(5t-1\right)^3X^3+t^4\left(5t+4\right)^2\left(t-1\right)^4\left(5t-1\right)^4X^4 \;=\; 0 \quad .$$

(This is not (yet, May 15, 2023) in the OEIS. Note that the straight enumeration version is A104184 of [Sl], `https://oeis.org/A104184`). For the sake of the OEIS, here are the first 30 terms:

$$0,0,3,18,113,636,3487,18656,98429,514012,2664690,13737758,70522801,360806214,1840913908,$$
$$9371761174,47621259557,241601881822,1224111502194,6195045902854,$$
$$31321134873744,158217553824544,798622703316154,4028438371631942,$$
$$20308239308212037,102323623873153810,515313296262175206,2594054240062008690,$$
$$13053194513626873348,65659889953142043376 \quad .$$

Strict Generalized Dyck paths

If we want to count strict generalized Dyck paths (respectively, sum of the areas), i.e. paths that never touch the x-axis except at the endpoints, use procedures `EqGFtS(S,X,t)` and `qEqGFtS(S,X,t)` respectively.

For the algebraic equation for the generating function for the sum of the areas under **strict** classical Dyck paths, type

`qEqGFtS({1,-1},X,t);`, getting that the equation is

$$\left(4t^2-1\right)X+t^2 \;=\; 0 \quad ,$$

that implies

$$X(t)=\frac{t^2}{1-4t^2} \quad ,$$

confirming, *purely automatically*, the following elegant proposition first discovered, and proved, in [SRW] (see also [C]):

Proposition (Shapiro, Rogers, and Woan) The sum of the areas of the strict Dyck paths of length $2n$ is 4^{n-1}.

What about sum-of-the-areas of strict Motzkin paths? Typing

`qEqGFtS({ 1 ,0,-1 },X,t);` gives

$$\left(3t^2+2t-1\right)X+t^2 \,=\, 0 \quad ,$$

implying that

$$X(t)=\frac{t^2}{1-2t-3t^2} \quad .$$

This is A015518[n-1] of [Sl] (see `https://oeis.org/A015518`). This sequence has numerous combinatorial interpretations, but so far, the connection to the sum of the areas under strict Motzkin paths escaped notice.

Efficient Computation of many terms

It is well-known and fairly easy to see (e.g. the modern classic [KP]) that every algebraic formal power series is D-finite, and equivalently, the sequence itself satisfies some **linear recurrence equation with polynomial coefficients**. The Maple package `gfun` (designed by Bruno Salvy and Paul Zimmermann) can do it for you, but it is just as easy to use *guessing*. Using *dynamical programming* one can easily crank out many terms, and then use 'guessing' (implemented in `listtorec` in `gfun`, but we have our own home-made version). Now we can even talk about sum-of-square-of-areas, and even higher powers.

Sample Output

Straight Enumeration of Generalized Dyck Paths

• If you want to see 16 theorems stating algebraic equations satisfied by generating functions enumerating generalized Dyck paths with set of steps that are subsets of $\{-2,-1,0,1,2\}$ (excluding trivial cases) and sometimes linear recurrences with polynomial coefficients satisfied by the sequences themselves, enjoy

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oGDW1.txt` .

• If you want to see 94 theorems stating algebraic equations satisfied by generating functions enumerating generalized Dyck paths with set of steps that are subsets of $\{-3,-2,-1,0,1,2,3\}$ (excluding trivial cases) and sometimes linear recurrences with polynomial coefficients satisfied by the sequences themselves, enjoy

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oGDW2.txt` .

[Of course this file includes all the theorems in the previous file].

• If you want to see (one) theorem about the algebraic equation satisfied by the generating function enumerating generalized Dyck paths with set of steps $\{-4,-3,-2,-1,0,1,2,3,4\}$, look at

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oGDW3.txt` .

It has degree 41 in $X(t)$!

Enumeration According to the Sum of Areas of Generalized Dyck Paths

• If you want to see 16 theorems stating algebraic equations for the SUM OF THE AREAS under generalized Dyck paths for all non-trivial subsets of $\{-2,-1,0,1,2\}$, as well, as estimates for the asymptotics of the average area (divided by $n^{3/2}$ (the ratio always tends to some constant)), see

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oGDW7.txt` .

• If you want to see interesting information about generalized Dyck paths for all non-trivial subsets of $\{-2,-1,0,1,2\}$, that consist of linear recurrences (obtained by guessing, but definitely correct, since we know they exist from theoretical reasons), for the straight enumeration, enumeration by 'sum of areas', and sometimes, enumeration by 'sum of area-squared', enabling estimates not only of the asymptotic average area, but also of the asymptotic variance, look here:

`https://sites.math.rutgers.edu/~zeilberg/tokhniot/oGDW8.txt` .

Conclusion

We described an interesting application of Bruno Buchberger's seminal Gröbner Basis algorithm, and at the same time demonstrated how an important class of enumerative combinatorics problems can be **fully automated**, using the great power of modern Computer Algebra Systems (in our case `Maple`), also using calculus (that Maple knows very well), as well as good-old numerical (and symbolic) dynamical programming. We also demonstrated how to teach the computer to build, *ab initio*, the system of equations (sometimes quite large) before asking it to kindly solve it.

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