The number of 1...d-avoiding permutations of length d+r for SYMBOLIC d but numeric r

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Dedicated to Ira Martin Gessel (b. April 9, 1951), on his millionth$_2$ birthday

Preface: How many permutations are there of length googol+30 avoiding an increasing subsequence of length googol?

This number is way too big for our physical universe, but the number of permutations of length googol+30 that *contain* at least one increasing subsequence of length googol is

3769987628815905643852921525646105664146833823621994801456991357113502936781270538054719048039675278
0769193354371721350001524610578097700045972792823897290959624203896101981952929640805170129282073883
4740018807571147534091229951249435913149171795302592312477456091277812321956212802204785578598020255
5625008802850838455586257402947256848380647181479993222566420025908679106917004348077812428261510240
6340176300585397517990032393036653951304924586489968650809789292291489270968710994809677050176596751
0725956202350750841376095024046396844968511243494784162014881795337835528626142808150073111101283361
0980701571937952824136796425017224636196853995950587943259043687431653922927840572864396105085190223
2582799067810378389890635196322425667467335158890500820128768331750855469963050322432973194472331947
0989825934469696079344723053679001130033667827524966034661782064851068214182454731365743413486729730
0631055444127725930013792836515384850702346797298406803049230145697433567004811555984158378611125895
0145768901348725550726037527669812626353266837685037397408862767082018239579392663024131792105407280
4788720840618514463465035392103884394981202007834724110094447116613439128758285044269471808502083275
6629374247928521501786839409853287740758570056230853738462527374534709641735458487560816949365616486
0695626913029699922648102091615529949414940648588048836485372758775808743231365617459515329190972398
7074543946415578728439994306071279654045146432379557541358408978156863172980419720839292761025261752
6805876626590163265795592248178664681630980893821587688413815206609216082514983787883386977226071420
21649147728993592578961422177700294482596740993986519357246959930668146505085270714475561150113747221
20887870047753358177316206263356927955729458754686550644432634687680282027976402772772483836117105473
48145611509228154510472000404130614639780926417137329939732465722014680564902839930824306834920414545
13874753600055252092001136814571329384587325582468487887244395295245585419188646792764252832159962029
69411649544372131053235387439446875434698793735121412796400236965732584484687219982898355145980291977
86269234486135973112564250247007239135280355775712267954726019033893771378762777423669196575295174512
96452587669725726144832740371782822308006170531910099265678141483622517144014107716217010098383839968
84507804590244720667406593929564134591547805793634468523934454328906756758701209765471514880572370759
09084331736216302289075177161806402089083889989467334293366576755423738845099552628279269937176915588
59427835870444539800644480052821630922331779937023228656305272974159931977365064817849361860909445301
08120140577436690007140701570599484176861047461052826774744899246746666909264578067076243923453088561
96698778069217767194382941365732112039412879713531991598317675682505439845424625600438225076973116586
49130213308514799728830764637172129004065611907475610401713008790972891409020362658741946509891832165
77016676670060012096109989093803820108650038852207775655317011335432185883307209708526943588264818977

377573814918607368593458655828559663290163681887888604288332683913232705939130899015285775019180974563
487912142476276560621310123468845009650614775925658273562207923751954793994347093016618292164580402712
542798143388641167614178301190598174793878806944301625322109937912755282207791779022466004479258408

(see below for line-accurate transcription) where the product is over all the n cells c of the Young diagram and $h(c)$ is its *hook length.* If $c=(i,j)$, then $h(c)=(\lambda_i-i)+(\lambda'_j-j)+1$, where λ' is the *conjuagate* diagram, where the rows become columns and vice-versa.

Let r be a fixed integer, then for *symbolic* d, valid for $d \geq r-1$, any Young diagram with $d+r$ cells can be written, for some Young diagram μ with $\leq r$ cells as $\mu=(\mu_1,\dots,\mu_r)$ (where we add zeros to the end if the number of parts of μ is less than r)

$$\lambda=(1+\mu_1,\dots,1+\mu_r,1^{d-r+r'}) \quad ,$$

where $r'=r-|\mu|$. For such a λ, with at least d rows,

$$\prod_{c\in\lambda} h(c)=\left(\prod_{c\in\mu} h(c)\right)\cdot((d+r'+\mu_1)(d+r'-1+\mu_2)\cdots(d+r'-r+1+\mu_r))\cdot(d-r+r')! \quad .$$

Hence f_λ, that is $(d+r)!$ divided by the above, is a certain specific number times a certain polynomial in d. Since, for s specific, *numeric*, r, there are only *finitely* many Young diagrams with at most r cells, the computer can find all of them, compute the polynomial corresponding to each of them, square it, and add-up all these terms, getting an *explicit* **polynomial** expression, in the variable d, for $B_d(d+r)$, the number of permutations of length $d+r$ that *contain* an increasing subsequence of length d. As we said above, from this we can find $G_d(d+r)=(d+r)!-B_d(d+r)$, valid for *symbolic* d larger than $r-1$.

$\mathbf{B_d(d+r)}$ for r from 1 to 30

$$B_d(d+1)=d^2+1 \quad ,$$

$$B_d(d+2)=\frac{1}{2}\,d^4+d^3+\frac{1}{2}\,d^2+d+3 \quad ,$$

$$B_d(d+3)=\frac{1}{6}\,d^6+d^5+\frac{5}{3}\,d^4+\frac{2}{3}\,d^3+\frac{19}{6}\,d^2+\frac{31}{3}\,d+11 \quad ,$$

$$B_d(d+4)=47+\frac{1}{24}\,d^8+\frac{1}{2}\,d^7+\frac{395}{6}\,d+\frac{247}{6}\,d^2+\frac{29}{24}\,d^4+9\,d^3+\frac{25}{12}\,d^6+\frac{19}{6}\,d^5 \quad ,$$

$$B_d(d+5)=239+\frac{1}{120}\,d^{10}+\frac{1}{6}\,d^9+\frac{31}{24}\,d^8+\frac{14}{3}\,d^7+\frac{3981}{10}\,d+\frac{10459}{30}\,d^2+\frac{653}{24}\,d^4+\frac{959}{6}\,d^3+\frac{823}{120}\,d^6+\frac{67}{30}\,d^5 \quad .$$

For $B_d(d+r)$ for r from 6 up to 30, see
`http://www.math.rutgers.edu/~zeilberg/tokhniot/oGessel64a` .

Sequences

The sequence $G_3(n)$ is the greatest *celeb* in the kingdom of combinatorial sequences, the super-famous **A000108** in Neil Sloane's legendary database ([Sl]). $G_4(n)$ is also moderately famous, **A005802**. $G_5(n)$ is **A047889**, $G_6(n)$ is **A047890**, $G_7(n)$ is **A052399** , $G_8(n)$ is **A072131**, $G_9(n)$ is **A072132**, $G_{10}(n)$ is **A072133**, $G_{11}(n)$ is **A072167**, But $G_d(n)$ for $d \geq 12$ are no longer there (for a good reason, enough is enough!). Using the polynomials $B_d(d+r)$, we computed the first

$2d+1$ terms of $G_d(n)$ for $d \leq 30$. See
`http://www.math.rutgers.edu/~zeilberg/tokhniot/oGessel64b` .

But, this method can only go up to $2d+1$ terms of the sequence $G_d(n)$, and of course, the first d terms are trivial, namely $n!$. Can we find the first 100 terms (or whatever) for the sequences $G_d(n)$ for d up to 20, and beyond, **efficiently?**

Encore: Efficient Computer-Algebra Implementation of Ira Gessel's AMAZING Determinant Formula

Recall Ira Gessel's [G] famous expression for the generating function of $G_d(n)$, *canonized* in the *bible* ([W], p. 996, Eq. (5)). Here it is:

$$\sum_{n \geq 0} \frac{G_d(n)}{n!^2} x^{2n} = \det(I_{|i-j|}(2x))_{i,j=1,\ldots,d} \quad ,$$

in which $I_\nu(t)$ is (the modified Bessel function)

$$I_\nu(t) = \sum_{j=0}^{\infty} \frac{(\frac{1}{2}\,t)^{2j+\nu}}{j!(j+\nu)!} \quad .$$

Can we use this to compute the first 100 terms, of say, $G_20(n)$?

While computing *numerical* determinants is very fast, computing *symbolic* ones is a different story. First, do not get scared by the "infinite" power series. If we are only interested in the first N terms of $G_d(n)$, then it is safe to truncate the series up to t^{2N}, and take the determinant of a $d \times d$ determinants with *polynomial entries*. If you use the *vanilla* determinant in a computer-algebra system such as Maple, it would be very inefficient, since the degree of the determinant is much larger than $2N$. But a little of cleverness can make things more efficient. The Maple package `Gessel64`, available free of charge from

`http://www.math.rutgers.edu/~zeilberg/tokhniot/Gessel64`

accompanying this article, has a procdure `SeqIra(k,N)` that computes the first N terms of $G_k(n)$, using the following clever implementation of Gessel's famous formula

```
SeqIra:=proc(k,N) local gu,t,i,j, R:

R := table():

R['0'] := 0:

R['1'] := 1:

R['+'] := '+':

R['-'] := '-':
```

```
R[`*`] := proc(p, q):  return add(coeff(p*q, t, i)*t**i, i=0..2*N): end:

R[`=`] := proc(p, q):  return evalb(p = q):  end:

gu:=expand(LinearAlgebra[Generic][Determinant][R](Matrix([seq([seq(Iv(abs(i-j),t,2*N),j=1..k-1)],i=1..k-1)]))):

[seq(coeff(gu,t,2*i)*i!**2,i=1..N)]:

end:
```

In the above code, procedure `Iv(v,t,N)` computes the truncated modified Bessel function that shows up in Gessel's determinant, and it is short enough to reproduce here.

```
Iv:=proc(v,t,N) local j:

add(t**(2*j+v)/j!/(j+v)!,j=0..trunc((N-v)/2)+1):

end:
```

Using this procedure, the first-named author computed (in 4507 seconds) the first 100 terms of each of the sequences $G_d(n)$ for $3 \le d \le 20$, and could have gone much further.

See `http://www.math.rutgers.edu/~zeilberg/tokhniot/oGessel64c` .

HAPPY 64th BIRTHDAY, IRA!

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