

The number of 1...d-avoiding permutations of length  $d+r$  for SYMBOLIC  $d$  but numeric  $r$

By Shalosh B. EKHAD, Nathaniel Shar, and Doron ZEILBERGER

*Dedicated to Ira Martin Gessel (b. April 9, 1951), on his millionth<sub>2</sub> birthday*

**Preface: How many permutations are there of length googol+30 avoiding an increasing subsequence of length googol?**

This number is way too big for our physical universe, but the number of permutations of length googol+30 that *contain* at least one increasing subsequence of length googol is

3769987628815905643852921525646105664146833823621994801456991357113502936781270538054719048039675278  
0769193354371721350001524610578097700045972792823897290959624203896101981952929640805170129282073883  
4740018807571147534091229951249435913149171795302592312477456091277812321956212802204785578598020255  
5625008802850838455586257402947256848380647181479993222566420025908679106917004348077812428261510240  
6340176300585397517990032393036653951304924586489968650809789292291489270968710994809677050176596751  
0725956202350750841376095024046396844968511243494784162014881795337835528626142808150073111101283361  
0980701571937952824136796425017224636196853995950587943259043687431653922927840572864396105085190223  
2582799067810378389890635196322425667467335158890500820128768331750855469963050322432973194472331947  
0989825934469696079344723053679001130033667827524966034661782064851068214182454731365743413486729730  
0631055444127725930013792836515384850702346797298406803049230145697433567004811555984158378611125895  
0145768901348725550726037527669812626353266837685037397408862767082018239579392663024131792105407280  
4788720840618514463465035392103884394981202007834724110094447116613439128758285044269471808502083275  
6629374247928521501786839409853287740758570056230853738462527374534709641735458487560816949365616486  
0695626913029699922648102091615529949414940648588048836485372758775808743231365617459515329190972398  
7074543946415578728439994306071279654045146432379557541358408978156863172980419720839292761025261752  
6805876626590163265795592248178664681630980893821587688413815206609216082514983787883386977226071420  
21649147728993592578961422177700294482596740993986519357246959930668146505085270714475561150113747221  
20887870047753358177316206263356927955729458754686550644432634687680282027976402772772483836117105473  
48145611509228154510472000404130614639780926417137329939732465722014680564902839930824306834920414545  
13874753600055252092001136814571329384587325582468487887244395295245585419188646792764252832159962029  
69411649544372131053235387439446875434698793735121412796400236965732584484687219982898355145980291977  
86269234486135973112564250247007239135280355775712267954726019033893771378762777423669196575295174512  
96452587669725726144832740371782822308006170531910099265678141483622517144014107716217010098383839968  
84507804590244720667406593929564134591547805793634468523934454328906756758701209765471514880572370759  
09084331736216302289075177161806402089083889989467334293366576755423738845099552628279269937176915588  
59427835870444539800644480052821630922331779937023228656305272974159931977365064817849361860909445301  
08120140577436690007140701570599484176861047461052826774744899246746666909264578067076243923453088561  
96698778069217767194382941365732112039412879713531991598317675682505439845424625600438225076973116586  
49130213308514799728830764637172129004065611907475610401713008790972891409020362658741946509891832165  
77016676670060012096109989093803820108650038852207775655317011335432185883307209708526943588264818977

37757381491860736859345865582855966329016368188788860428833268391323270593913089901528577501918097456  
34879121424762765606213101234688450096506147759256582735622079237519547939943470930166182921645804027  
12542798143388641167614178301190598174793878806944301625322109937912755282207791779022466004479258408  
24462949592761349881316543422038699183826473525107075809508274778093413168220963984409028566362293900  
40215415882419358649518674355414801095047413826040824566345129789426039221842088797052981439573736696  
50223307930886494490895506612422266377009758720488022558779513425100432348926430427665125944987693089  
42245751122706284028982754337386885459391626543570555162051612664363788373280457226691660908679569539  
27163081562519904030045933274931742332018704568957075002591805894557106029373427199758644919233869688  
59038428977769800021296515522194835877177597740437988812991749584835721786755293502620149338987031222  
32518225184081589902714463624365018242747599082635817593737724580337688809342550695342366935036425354  
91880914435376674876432270204764414065561382212425100253695336680109353578878041405262772638139124792  
83216406483941960286265199599663254512526642623538896318838417766536461292705936611493062590853978024  
18629266233934211681736693714241352634384615108485320700947811487618744149158225668175169324385259284  
55634363409372944818437842421507459176260334046758894630063276039591166623100092626550628336007090706  
43413326647797799377122631843882036054772111620148059377505229785356202259250047229187386576746999449  
47405347907659143618050579417087497652165460185477043345636632204978226001800424273526341460220242548  
683728799179065030083029494514450905531725089967903293290935500874548539339178735194085694882107486318  
79883374585250820777287677645800280443076699166062637606763797770235404212193344610052823762990072265  
783070820234545141480898874637486106893816774598214664007156038886731975384257202382 .

Hence the number of permutations of length googol+30 *avoiding* an increasing subsequence of length googol is  $(\text{googol} + 30)!$  *minus* the above small number.

### Counting the “Bad Guys”

Recall that thanks to Robinson-Schenstead ([R][Sc]), the number of permutations of length  $n$  that **do not** contain an increasing sequence of length  $d$  is given by

$$G_d(n) := \sum_{\substack{\lambda \vdash n \\ \# \text{rows}(\lambda) < d}} f_\lambda^2 ,$$

where  $\lambda$  denotes a typical *Young diagram*.

Hence the number of permutations of length  $n$  that **do** contain an increasing sequence of length  $d$  is

$$B_d(n) := \sum_{\substack{\lambda \vdash n \\ \# \text{rows}(\lambda) \geq d}} f_\lambda^2 .$$

Since the total number of permutations of length  $n$  is  $n!$  ([B]), if we know how to find  $B_d(n)$ , we would know immediately  $G_d(n) = n! - B_d(n)$ , at least if we leave  $n!$  alone as a factorial, rather than spell it out.

Recall that the *Hook Length formula* (see [Wiki]) tells you that if  $\lambda$  is a Young diagram then

$$f_\lambda = \frac{n!}{\prod_{c \in \lambda} h(c)} ,$$

where the product is over all the  $n$  cells  $c$  of the Young diagram and  $h(c)$  is its *hook length*. If  $c = (i, j)$ , then  $h(c) = (\lambda_i - i) + (\lambda'_j - j) + 1$ , where  $\lambda'$  is the *conjugate* diagram, where the rows become columns and vice-versa.

Let  $r$  be a fixed integer, then for *symbolic*  $d$ , valid for  $d \geq r - 1$ , any Young diagram with  $d + r$  cells can be written, for some Young diagram  $\mu$  with  $\leq r$  cells as  $\mu = (\mu_1, \dots, \mu_r)$  (where we add zeros to the end if the number of parts of  $\mu$  is less than  $r$ )

$$\lambda = (1 + \mu_1, \dots, 1 + \mu_r, 1^{d-r+r'}) \quad ,$$

where  $r' = r - |\mu|$ . For such a  $\lambda$ , with at least  $d$  rows,

$$\prod_{c \in \lambda} h(c) = \left( \prod_{c \in \mu} h(c) \right) \cdot ((d + r' + \mu_1)(d + r' - 1 + \mu_2) \cdots (d + r' - r + 1 + \mu_r)) \cdot (d - r + r')! \quad .$$

Hence  $f_\lambda$ , that is  $(d+r)!$  divided by the above, is a certain specific number times a certain polynomial in  $d$ . Since, for s specific, *numeric*,  $r$ , there are only *finitely* many Young diagrams with at most  $r$  cells, the computer can find all of them, compute the polynomial corresponding to each of them, square it, and add-up all these terms, getting an *explicit polynomial* expression, in the variable  $d$ , for  $B_d(d+r)$ , the number of permutations of length  $d+r$  that *contain* an increasing subsequence of length  $d$ . As we said above, from this we can find  $G_d(d+r) = (d+r)! - B_d(d+r)$ , valid for *symbolic*  $d$  larger than  $r - 1$ .

### **$B_d(d+r)$ for $r$ from 1 to 30**

$$B_d(d+1) = d^2 + 1 \quad ,$$

$$B_d(d+2) = \frac{1}{2} d^4 + d^3 + \frac{1}{2} d^2 + d + 3 \quad ,$$

$$B_d(d+3) = \frac{1}{6} d^6 + d^5 + \frac{5}{3} d^4 + \frac{2}{3} d^3 + \frac{19}{6} d^2 + \frac{31}{3} d + 11 \quad ,$$

$$B_d(d+4) = 47 + \frac{1}{24} d^8 + \frac{1}{2} d^7 + \frac{395}{6} d + \frac{247}{6} d^2 + \frac{29}{24} d^4 + 9 d^3 + \frac{25}{12} d^6 + \frac{19}{6} d^5 \quad ,$$

$$B_d(d+5) = 239 + \frac{1}{120} d^{10} + \frac{1}{6} d^9 + \frac{31}{24} d^8 + \frac{14}{3} d^7 + \frac{3981}{10} d + \frac{10459}{30} d^2 + \frac{653}{24} d^4 + \frac{959}{6} d^3 + \frac{823}{120} d^6 + \frac{67}{30} d^5 \quad .$$

For  $B_d(d+r)$  for  $r$  from 6 up to 30, see

<http://www.math.rutgers.edu/~zeilberg/tokhniot/oGessel64a> .

### **Sequences**

The sequence  $G_3(n)$  is the greatest *celeb* in the kingdom of combinatorial sequences, the super-famous **A000108** in Neil Sloane's legendary database ([Sl]).  $G_4(n)$  is also moderately famous, **A005802**.  $G_5(n)$  is **A047889**,  $G_6(n)$  is **A047890**,  $G_7(n)$  is **A052399**,  $G_8(n)$  is **A072131**,  $G_9(n)$  is **A072132**,  $G_{10}(n)$  is **A072133**,  $G_{11}(n)$  is **A072167**, But  $G_d(n)$  for  $d \geq 12$  are no longer there (for a good reason, enough is enough!). Using the polynomials  $B_d(d+r)$ , we computed the first

$2d + 1$  terms of  $G_d(n)$  for  $d \leq 30$ . See  
<http://www.math.rutgers.edu/~zeilberg/tokhniot/oGessel64b> .

But, this method can only go up to  $2d + 1$  terms of the sequence  $G_d(n)$ , and of course, the first  $d$  terms are trivial, namely  $n!$ . Can we find the first 100 terms (or whatever) for the sequences  $G_d(n)$  for  $d$  up to 20, and beyond, **efficiently**?

**Encore: Efficient Computer-Algebra Implementation of Ira Gessel’s AMAZING Determinant Formula**

Recall Ira Gessel’s [G] famous expression for the generating function of  $G_d(n)$ , *canonized* in the *bible* ([W], p. 996, Eq. (5)). Here it is:

$$\sum_{n \geq 0} \frac{G_d(n)}{n!^2} x^{2n} = \det(I_{|i-j|}(2x))_{i,j=1,\dots,d} \quad ,$$

in which  $I_\nu(t)$  is (the modified Bessel function)

$$I_\nu(t) = \sum_{j=0}^{\infty} \frac{(\frac{1}{2}t)^{2j+\nu}}{j!(j+\nu)!} \quad .$$

Can we use this to compute the first 100 terms, of say,  $G_{20}(n)$ ?

While computing *numerical* determinants is very fast, computing *symbolic* ones is a different story. First, do not get scared by the “infinite” power series. If we are only interested in the first  $N$  terms of  $G_d(n)$ , then it is safe to truncate the series up to  $t^{2N}$ , and take the determinant of a  $d \times d$  determinants with *polynomial entries*. If you use the *vanilla* determinant in a computer-algebra system such as Maple, it would be very inefficient, since the degree of the determinant is much larger than  $2N$ . But a little of cleverness can make things more efficient. The Maple package **Gessel64**, available free of charge from

<http://www.math.rutgers.edu/~zeilberg/tokhniot/Gessel64>

accompanying this article, has a procedure **SeqIra(k,N)** that computes the first  $N$  terms of  $G_k(n)$ , using the following clever implementation of Gessel’s famous formula

```
SeqIra:=proc(k,N) local gu,t,i,j, R:
R := table():
R['0'] := 0:
R['1'] := 1:
R['+'] := '+':
R['-'] := '-':
```

```

R['*'] := proc(p, q): return add(coeff(p*q, t, i)*t**i, i=0..2*N): end:

R['='] := proc(p, q): return evalb(p = q): end:

gu:=expand(LinearAlgebra[Generic][Determinant][R](Matrix([seq([seq(Iv(abs(i-j),t,2*N),j=1..k-1)],i=1..k-1])))):

[seq(coeff(gu,t,2*i)*i!**2,i=1..N)]:

end:

```

In the above code, procedure  $Iv(v,t,N)$  computes the truncated modified Bessel function that shows up in Gessel's determinant, and it is short enough to reproduce here.

```

Iv:=proc(v,t,N) local j:

add(t**(2*j+v)/j!/(j+v)!,j=0..trunc((N-v)/2)+1):

end:

```

Using this procedure, the first-named author computed (in 4507 seconds) the first 100 terms of each of the sequences  $G_d(n)$  for  $3 \leq d \leq 20$ , and could have gone much further.

See <http://www.math.rutgers.edu/~zeilberg/tokhniot/oGessel64c> .

## HAPPY 64th BIRTHDAY, IRA!

### References

- [B] Rabbi Levi Ben Gerson, *Sefer Maaseh Hoshev*, Avignon, 1321.
- [G] I. Gessel, *Symmetric functions and P-recursiveness*, Journal of Combinatorial Theory, Series A **53** (1990), 257-285;  
<http://people.brandeis.edu/~gessel/homepage/papers/dfin.pdf> .
- [R] G. de B. Robinson, *On the representations of  $S_n$* , Amer. J. Math. **60** (1938), 745-760.
- [Sc] C. E. Schensted, *Largest increasing and decreasing subsequences*, Canad. J. Math **13** (1961), 179-191.
- [SI] N.J.A. Sloane, *The On-Line Enclopedia of Integer Sequences*, <http://oeis.org> .
- [Wiki] The Wikipedia Foundation, *Hook Length Formula*, [http://en.wikipedia.org/wiki/Hook\\_length\\_formula](http://en.wikipedia.org/wiki/Hook_length_formula)
- [W] H. Wilf, *Mathematics, an experimental science*, in: "Princeton Companion to Mathematics", (W. Timothy Gowers, ed.), Princeton University Press, 2008, 991-1000;  
<http://www.math.rutgers.edu/~zeilberg/akherim/HerbMasterpieceEM.pdf> .

---

Shalosh B. Ekhad, c/o D. Zeilberger, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.

---

Nathaniel Shar, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.  
nshar at math dot rutgers dot edu ; <http://www.math.rutgers.edu/~nbs48/> .

---

Doron Zeilberger, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.  
zeilberg at math dot rutgers dot edu ; <http://www.math.rutgers.edu/~zeilberg/> .

---

Published in The Personal Journal of Shalosh B. Ekhad and Doron Zeilberger  
(<http://www.math.rutgers.edu/~zeilberg/pj.html>) and [arxiv.org](http://arxiv.org).

---

**March 27, 2015**