A SHORT WZ-STYLE PROOF OF ABEL'S IDENTITY

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Dedicated to Dominique Foata on the occasion of his sixtieth birthday.

Abstract. Using a certification procedure for Abel-type sums, we present a computerized proof of Abel's identity.

Abel's identity [1],[2] can be proved in many ways, including the elegant combinatorial methods of [3]. Here we present the first computer-generated proof using the methods introduced in [4].

Theorem([1]; [2], p.128). *For $n \geq 0$:*

$$\sum_{k=0}^{n} \binom{n}{k} (r+k)^{k-1}(s-k)^{n-k} = \frac{(r+s)^n}{r} \,. \tag{1}$$

Proof. Let $F_{n,k}(r,s)$ and $a_n(r,s)$ denote, respectively, the summand and sum on the LHS of (1), and let $G_{n,k} := (s-n)\binom{n-1}{k-1}(k+r)^{k-1}(s-k)^{n-k-1}$. Since

$$F_{n,k}(r,s) - sF_{n-1,k}(r,s) - (n+r)F_{n-1,k}(r+1,s-1) + (n-1)(r+s)F_{n-2,k}(r+1,s-1) = G_{n,k} - G_{n,k+1}\,,$$

(check!), we have by summing from $k=0$ to $k=n$, thanks to the telescoping on the right:

$$a_n(r,s) - sa_{n-1}(r,s) - (n+r)a_{n-1}(r+1,s-1) + (n-1)(r+s)a_{n-2}(r+1,s-1) = 0.$$

Since $(r+s)^n \cdot r^{-1}$ also satisfies this recurrence (check!) with the same initial conditions $a_0(r,s) = r^{-1}$ and $a_1(r,s) = (r+s)\cdot r^{-1}$, (1) follows. □

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References

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