The Binomial Theorem for $(N + n)^r$ (where Nf(n)=f(n+1))

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Added Dec. 13, 2011: The main theorm of this note is contained in Lemma 5 of "On Two-generated Non-commutative Algebras Subject to the Affine Relation" by Christoph Koutschan, Viktor Levandovskyy, Oleksandr Motsak, `http://arxiv.org/abs/1108.1108`, who prove many other results, and a stronger version of our main result (using Stirling numbers).

This note has only personal and historical interest, and is only published in the Personal Journal of Ekhad and Zeilberger and arxiv.org .

We all know the *binomial theorem*

$$(x + y)^r = \sum_{i=0}^{r} \frac{r!}{i!(r-i)!} x^i y^{r-i} \quad , \tag{1}$$

where x and y are *commuting* variables, i.e. $yx = xy$. The binomial theorem is easily proved by induction on r.

Not as famous is the *quantum* analog, that goes back to Marco Schützenberger,

$$(x + y)^r = \sum_{i=0}^{r} \frac{[r]!}{[i]![r-i]!} x^i y^{r-i} \quad , \tag{2}$$

where x and y are *q-commuting* variables, i.e. $yx = qxy$, and $[j]! := (1)(1 + q)(1 + q + q^2) \cdots (1 + q + \ldots + q^{j-1})$. The quantum binomial theorem is also easily proved by induction on r.

Even more obscure is the binomial theorem for $(D + x)^r$, where D is the differentiation operator $\frac{d}{dx}$ (so $Dx = xD + 1$):

$$(x + D)^r = \sum_{k=0}^{\lfloor r/2 \rfloor} \frac{r!}{2^k k!} \sum_{j+l=r-2k} \frac{1}{j!l!} x^j D^l \quad , \tag{3}$$

that is also easily proved by induction.

But nothing analogous is known for $(n + N)^r$, where N is the *shift operator* $Nf(n) := f(n + 1)$, and n is multiplication by n. Now the *commutation relation* is $Nn = nN + N$ and things seem to get messier.

Let's first try to expand $(N + n)^r$ in powers of N with coefficients that are polynomials in n:

$$(N + n)^r = \sum_{d=0}^{r} P_{r,d}(n) N^d \quad . \tag{4a}$$

We claim that the coefficients $P_{r,d}(n)$ are given by the following "explicit" formula

$$P_{r,d}(n) = \sum_{\substack{p_0+\ldots+p_d=r-d \\ 0\le p_0,\ldots p_d \le r-d}} \prod_{j=0}^{d} (n+j)^{p_j} \quad . \qquad (4b)$$

In other words, $P_{r,d}(n)$ is the *weight-enumerator* of *compositions* of $r-d$ into $d+1$ *non-negative* integers, with

$$Weight([p_0,\ldots,p_d]) := \prod_{j=0}^{d} (n+j)^{p_j} \quad .$$

Since $(N+n)^r = (N+n)(N+n)^{r-1}$ we have the recurrence:

$$P_{r,d}(n) = P_{r-1,d-1}(n+1) + nP_{r-1,d}(n) \quad .$$

Identity (4) is proved by induction by noting that any composition of $r-d$ into $d+1$ non-negative integers either has $p_0 = 0$ and beheading it yields a composition of $r-d$ into $(d-1)+1$ non-negative integers, and the weights get adjusted by replacing n by $n+1$, or $p_0 \ge 1$ and subtracting 1 from p_0 yields a composition of $r-d-1$ into $d+1$ non-negative integers, and adding the 1 back to the p_0 term results in multiplying the weight by n.

While formula (4) is very elegant and combinatorial, it would be nice to have an *explicit* formula, as a linear combination of monomials $n^i N^j$ analogous to (1), (2) and (3). This does not seem to be possible, but by using the Maple package `Nnr`, written by Doron Zeilberger, and downloadable from `http://www.math.rutgers.edu/~zeilberg/tokhniot/Nnr`, one can get an explicit formula for the first k highest-degree terms for any desired k, for $r \ge 2k$.

Let's describe the answer for $k = 3$. First we define,

$$((n+N))^r = \sum_{i=0}^{r} \frac{r!}{i!(r-i)!} n^i N^{r-i} \quad ,$$

in other words the polynomial in the (non-commuting) variables n and N obtained by expanding $(n+N)^m$ while pretending that n and N commute. We have (all the algebric computations below should be done in commutative algebra, and at the end each monomial should be expressed with n before N. i.e. in the form $n^i N^j$)

$$(N+n)^r = ((N+n))^r + \binom{r}{2} N((N+n))^{r-2} + \binom{r}{3} N((N+n))^{r-4} \left(\frac{1}{4}\,(3\,r-5)\,N + n\right)$$

$$+\binom{r}{4} N((N+n))^{r-6} \left(\frac{1}{2}\,(r-2)\,(r-3)\,N^2 + 2\,(r-3)\,nN + n^2\right) \quad +$$

$$(terms - of - total - degree \quad \le r-4) \quad .$$

To get all the terms of total degree $\ge r-10$, see `http://www.math.rutgers.edu/~zeilberg/tokhniot/oNnr10`. You are welcome to get all the terms of degree $\ge r-k$, for any desired positive integer k, by typing `Mispat(r,k,n,N);` in the Maple package `Nnr`.

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