Solutions of the Quiz for Doron Zeilberger's Lecture at Math Leagues Camp, TCNJ, July 23, 2018, 1:30-2:20pm

1. China played Soccer against Japan, and the score was

China: 5; Japan:4 .

How many possible game histories are there?

Sol. of 1:

$$\binom{5+4}{4} = \binom{9}{4} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 126 \quad .$$

2. Out of these game-histories , how many were **boring**? How many **exciting** ones?

Sol. of 2: Recall that a game is boring if the winning team was always **ahead**, i.e. there was not even a time when there was a temporary tie, like in the 2018 World Cup Final game between France and Croatia

$$FCFFFC \quad ,$$

where after the second goal both teams had one goal. In the lecture we proved that the number of such totally boring games, with score $(n+1, n)$ is

$$\frac{1}{2n+1}\binom{2n+1}{n} \quad .$$

In this problem $n = 4$, so the number of boring histories is

$$\frac{1}{2 \cdot 4+1}\binom{2 \cdot 4+1}{4} = \frac{1}{9}\binom{9}{4} = \frac{1}{9} \cdot 126 = 14 \quad .$$

Hence the number of **boring** game-histories is 14 and the number of exciting ones is $126 - 14 = 112$.

3. What 'bad' walk in the Manhattan lattice from $(1,0)$ to $(5,4)$ corresponds to the following walk from $(0,1)$ to $(5,4)$?

$$(0,1) \to (0,2) \to (1,2) \to (2,2) \to (3,2) \to (3,3) \to (4,3) \to (5,3) \to (5,4)$$

Sol. of 3: We look at the portion until it hits the diagonal (the line $y = x$) for the **first** time

$$(0,1) \to (0,2) \to (1,2) \to (2,2) \quad ,$$

and reflect it , by changing (a, b) to (b, a), getting

$$(1,0) \to (2,0) \to (2,1) \to (2,2) \quad ,$$

After that first encounter, we just keep it, as is, getting

$$(1,0) \to (2,0) \to (2,1) \to (2,2) \to (3,2) \to (3,3) \to (4,3) \to (5,3) \to (5,4) \quad .$$

4. What is the unique cyclic shift in the following game history between team A and team B, where team A scored 7 goals and team B scored 6 goals, that is boring.

$$AABBBABABAAAB \quad ,$$

Sol. of 4: After every goal, we write the **difference** between the number of As and the number of Bs (possibly a negative integer)

$$A(1)A(2)B(1)B(0)B(-1)A(0)B(-1)A(0)B(-1)A(0)A(1)A(2)B(1) \quad ,$$

(note that it must always end at 1, since there 7 As and 6 Bs). The **lowest** difference is -1, but it shows up **three** times, so we look at the **last** time it showed up, so the unique boring cyclic shift must start with $AAAB$ followed by the part that came in front of it.

The answer is

$$AAABAABBBABAB \quad .$$

5. In a circular track, there are stations where the amount of gasoline (in km) is indicated in **boldface**, and the distances between them are written in *italic*.

1 *3* **4** *4* **3** *3* **2** *6* **10** *4* ,

where would you start so that you will never run out of gas?

Sol. of 5: We first do a **dry run**, allowing for negative amount of gasoline in our gas tank.

At the start, the first station, we have gasoline for 1 km.

after traveling 3 km, right before the second station, we have -2 km of gasoline.

We add 4 km of gasoline, so now our gas tank has enough for $-2 + 4 = 2$ km.

We travel 4 km to the third station, and right before the third station, we have $2 - 4 = -2$ km of gasoline.

At the third station we add 3 km of gasoline, yielding $-2 + 3 = 1$ km of gasoline.

We travel 3 km to the fourth station, leaving us $1 - 3 = -2$ km of gasoline.

At the fourth station, we add the available 2 km of gasoline, leaving us $-2 + 2 = 0$ km of gasoline in our gas tank.

We travel 6 km to the fifth station, leaving us $0 - 6 = -6$ km of gasoline.

At the fifth station, we add the available 10 km of gasoline, leaving us $-6 + 10 = 4$ km of gasoline.

The distance between the fifth station to the starting point is 4 km, so we arrive safely with an empty gas tank.

But in real life, you can't have a negative amount of gasoline, so in order to be able to drive this track, we **start** at the station where, right before it, we had the **lowest amount**, in this problem it is the fifth (last) station.

So the answer is:

Start at the fifth station (where the amount of gasoline is 10). Now the track is

10 *4* **1** *3* **4** *4* **3** *3* **2** *6*

and you will never have a negative amount in your gas tank.