Comments on Doron Zeilberger's abstract

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Doron's abstract

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An Ultra-Finitistic Foundation of Probability
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Probability theory started on the right foot with Cardano, Fermat and Pascal when it restricted itself to finite sample spaces, and was reduced to counting finite sets. Then it got ruined by attempts to come to grips with that fictional (and completely superfluous) 'notion' they called 'infinity'.

A lot of probability theory can be done by keeping everything finite, and whatever can't be done that way, is not worth doing. We live in a finite world, and any talk of 'infinite' sample spaces is not even wrong, it is utterly meaningless. The only change needed, when talking about an 'infinite' sequence of sample spaces, say of n coin tosses, $\{H, T\}^n$, for 'any' n, tacitly implying that you have an 'infinite' supply of such n, is to replace it by the phrase 'symbolic n'.

This new approach is inspired by the philosophy and ideology behind symbolic computation. Symbolic computation can also redo, ab initio, without any human help, large parts of classical probability theory.

Comment 1: Probability before Cardano

The combinatorics of dice throwing was often taught in European universities from the 13th century onward. One resource was the Latin poem *De Vetula*, which explained how to count the 216 chances for three dice [1]. The anonymous author of *De Vetula* makes clear that each of the 216 chances has equal force and frequency.

Gamblers used these counts to fix stakes and bets (these are Laplace's words in the history that he placed at the end of his *Essai philosophique sur les probabilités*: *réglér les enjeux et paris* [9]). How? Historians have not located documents in which gamblers explained what they did, but it is easy enough to connect the dots. From at least the 11th century, both sides of the Mediterranean used commercial arithmetics that emphasized using the rule of three to solve every sort of proportionality problem, and once we can count chances, the rule of three lends itself to the solution of many probability problems [2].

Here is a simple question that gamblers educated in counting chances and the rule of three could easily answer.

Glenn's Simple Question 1. *Two dice are thrown repeatedly. Player A bets on 7 points and Player B bets on 8 points. In other words, A gets the money* *on the table if 7 comes up before 8, and B gets it if 8 comes up first. A puts 6 ecus on the table. How much should B put on the table?*

To answer the question we count the 36 chances for each throw.

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

We get 7 from 16, 25, 34, 43, 52, and 61. We get 8 from 26, 35, 44, 53, and 62. So the ratio of 7's force or frequency 8's force or frequency is 6 to 5; B wins only 5 times for every 6 times A wins. As A has put 6 ecus on the table for B to win, B will win 5×6 ecus while A is winning $6 \times$ (the amount B puts on the table). To make this fair, B must put 5 ecus on the table.

With a little more work, we can answer the more natural version of the question, where the players throw the dice alternately and win only when their own throw succeeds. We can also answer other popular questions, such as the following:

Glenn's Simple Question 2. *How many throws of a single die are needed to make it approximately an even bet that you will get a six?*

Glenn's Simple Question 3. *How much you should pay in order to win one ecu if you get 7 points on a single throw of three dice?*

Comment 2: Huygens introduces algebra.

Fermat was an early master of algebra, but the first document in which we see algebra deployed for probability problems is Christiaan Huygens's brief treatise *De rationciniis in ludo aleae*, first published in 1657.

We can obviously replace the rule of three with algebra in the reasoning I just spelled out: $5 \times 6 = 6 \times x$, so $x = 5$. Huygens uses algebra in a less trivial way in his last proposition, which I paraphrase as follows:

Huygens's Proposition XIV. *A and B take turns throwing two dice. A wins if he throws 7 points before B throws 6 points. If B throws first, what is the ratio of their chances?*

Whenever it is A's turn to throw, he has 6 chances out of 36 to win on that throw; whenever it is B's turn, he has 5 chances out of 36 to win on that throw. Huygens wrote a for the stakes for which they are playing; let us simplify by setting $a = 1$. Huygens wrote x for A's chance (*kans* in Dutch or *sors* or *expectatio* in Latin, by this Huygens meant the player's prospect or the value this prospect — the share of the stakes to which the player was entitled) at the outset and y for his chance if and when he gets to throw again, after B has lost his first throw. At the outset, B has 5 chances to win and 31 chances to put A in the position where his chance is y. So

$$x = \frac{5}{36} \times 0 + \frac{31}{36} \times y.$$

If B loses his first throw, then A has 6 chances of winning the 1 ecu on the table and 5 chances of returning to x. So

$$y = \frac{6}{36} \times 1 + \frac{30}{36} \times x.$$

Solving the two equations, we find that $x = \frac{30}{61}$. So the ratio of A's chance to B's is 30 to 31.

Comment 3: Fermat makes it harder.

Pascal and Fermat had corresponded in 1654. Neither their correspondence nor Pascal's treatise on using the arithmetic triangle to solve the division problem were published at the time, but Huygens learned something about what they had done when he visited in Paris in 1655. In 1656, Huygens drafted his treatise and shared the draft with both Fermat and Pascal through their friend Carcavy. They responded by posing more difficulty problems. In particular, Fermat proposed a variant on Huygens's Proposition XIV, which Huygens included as the first of five problems given without solutions at the end of his treatise.

Huygens's Problem 1. *A and B take turns throwing two dice. A wins if he throws 7 points before B throws 6 points. A goes first with a single throw. Then B gets 2 successive throws. Then A gets 2 successive throws. Then they continue alternating, each getting 2 throws each time until one of them wins. What is the ratio of their chances? Answer: 10355 to 12276.*

Although Huygens did not give a solution in his treatise, he had promptly sent a solution to Carcavy, in a letter dated July 6, 1656. This solution used, of course, the same algebraic method Huygens had used for his Proposition XIV.

Comment 4: Bernoulli makes it yet harder.

In the *Journal des Sçavans* in 1685, Jacob Bernoulli published without solution two even more difficult variants on the problem of alternating play [6, volume 1, pp. 19–24]. We can paraphrase the first as follows:

Bernoulli's Problem 1. *A and B take turns throwing a single die. The first to throw a 1 wins. A throws once, then B once. Then A throws twice, then B twice. Then A throws three times, then B three times. And so on until one of them wins. Find the ratio of their chances.*

The second was similar; A throws once, then B twice, then A three times, etc.

As Bernoulli explained in his commentary on Huygens's treatise in Part I of his *Ars conjectandi*, published posthumously 1713, these problems illustrate the difficulty in applying Huygens's algebraic method when the same situations do not repeatedly come back in the game and hence the chances do not repeat themselves, "but instead other new ones are produced, one after the other, different from the previous ones and equally unknown, *ad infinitum*" [5, pp. 175].

Bernoulli's method for solving problems of this type was to imagine that each throw is made by a different player and to calculate the value of each of these players' chances. To get back to the original formulation of the problem, we suppose that each of this infinitude of players gives his chance back to A or to B, and so the problem of finding the value of A's or B's total chance reduces to summing an infinite series. It can be summed numerically if necessary. (Here I continue to use "chance" to mean not probability but the player's prospect or its monetary value. Bernoulli, following the translation of Huygens's treatise from Dutch to Latin by his teacher van Schooten, used the Latin *expectatio* and *sors* with this meaning. In Dutch, Huygens had used *kans*, both in this meaning and for the individual equally easy case or chance.)

Bernoulli's 1685 challenge problem remained unanswered for five years. So in 1690, in Leibniz's *Acta Eruditorum*, he gave the infinite series that needed to be summed [4, pp. 91–93]. He gave numerical answers for some similar questions in *Ars conjectandi*.

Not everyone was comfortable with Bernoulli's actual, as opposed to potential, infinitude of players. Abraham De Moivre, for example, proposed a different argument. In his *De mensura sortis*, published in 1711 [8], and in his *Doctrine of Chances*, published in 1718 [7, p. vii], De Moivre justified using infinite sums, even in simpler problems like that of Huygens's Proposition XIV, by supposing that on each throw the player about to make it renounces his right to do so and is compensated by receiving the value of the throw from the money on the table. As the players proceed through the throws, never actually making them, the money on the table dwindles to insignificance, and the money held by each player approaches as closely as we wish the sum of the corresponding infinite series.

References

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