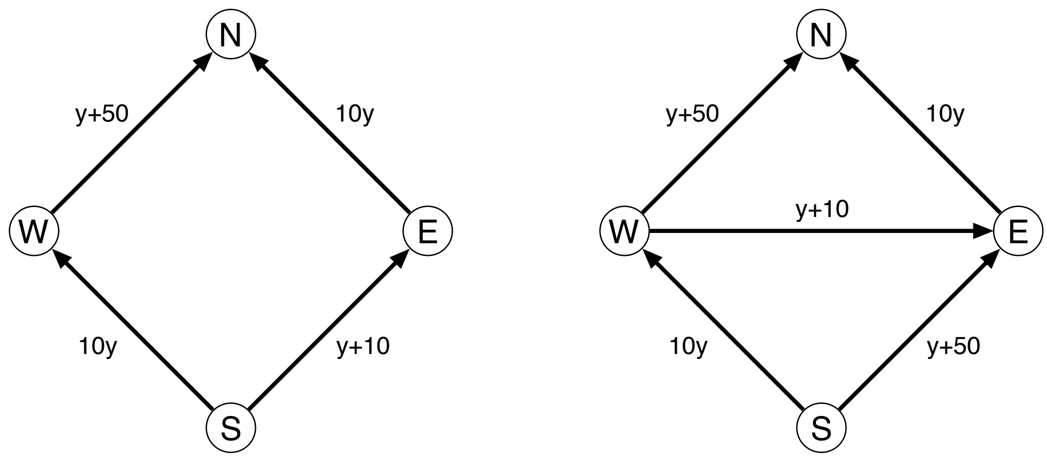
$25 Reward: Dr. Z's Challenge Problem

Suppose that 6 cars want to travel from node "S" to node "N" using the roads on the graphs below.



The labels on each edge show how long it takes to travel along the road when y cars are using it. For example, if two cars take the route $S \to W \to N$ and the other four take the route $S \to E \to N$, then it will take the first two cars $20 + 52 = 72$ minutes each and it will take the other four cars $54 + 40 = 92$ minutes each.

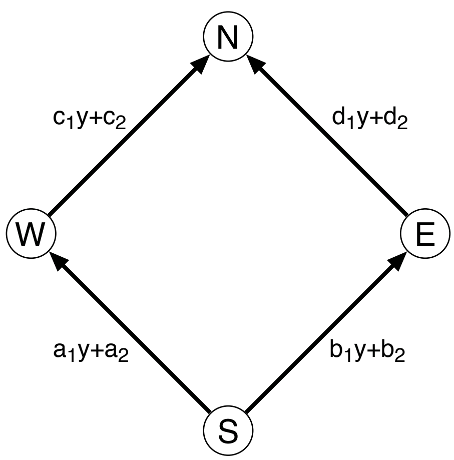
One of the four cars traveling along the east route may notice that it would be faster for her (individually) to go along the west route. If she re-routes, then there are now three cars along both routes and it will take everyone 83 minutes to go from S to N.

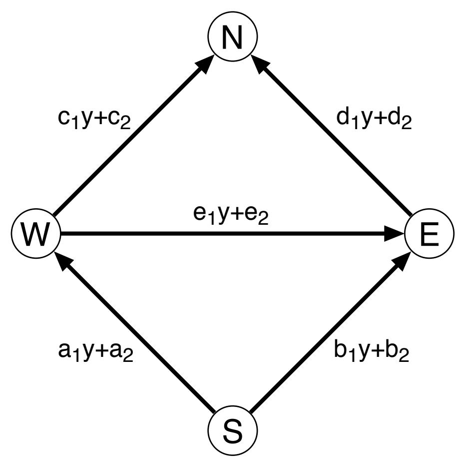
If we are using the road configuration on the left side of the figure, then this is the equilibrium: no car has any incentive to change their route. Each one will continue driving along the same route every day, always taking 83 minutes to do so.

However, if we are using the roads on the right side of the figure (so we add one very fast road $W \to E$), then someone on the west route may now notice that switching to the route $S \to W \to E \to N$ is advantageous. If just one car does this and everyone else stays as they are, then he will have a travel time of $30 + 11 + 40 = 71$ minutes, the two remaining cars on the west route will have a travel time of $30+52 = 82$ minutes, and the three cars taking the east route will have a travel time of $53 + 40 = 93$ minutes. Now other cars also have an incentive to change their route. This sets off a chain reaction, but the drivers quickly settle into a new equilibrium: two cars take each of the routes $S \to W \to N$, $S \to W \to E \to N$, and $S \to E \to N$. The resulting average travel time is 92 minutes.

Moral of the story: Adding choices does not always improve outcomes, and can actually make things worse! Adding the road $W \to E$ raised the average travel time from 83 to 92 minutes. (In this case, it actually made things worse for every car individually!) This surprising fact is known as Braess's Paradox.

Challenge: For arbitrary coefficients a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2, and n cars, we can consider the equilibria for each of the two road configurations below. Find coefficients and n so that the ratio of the average equilibria times (right/left) is larger than 92/83 - the larger the better. If possible do this for $n = 6$.





Tool for Experimentation: You can try out different scenarios using Dr. Z's Braess package for Maple, which you can download from the page linked below. Read in the package and use the command "ezra()" for instructions.

`https://sites.math.rutgers.edu/~zeilberg/programs.html`