SOLUTION TO PROBLEM #11850 OF THE MONTHLY

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Problem #11850. *Proposed by Zafar Ahmed, Bhabha Atomic Research Center, Mumbai, India.* Let A_n be the function given by

$$A_n(x) = \sqrt{\frac{2}{\pi}}\,\frac{1}{n!}\,(1+x^2)^{n/2}\frac{d^n}{dx^n}\left(\frac{1}{1+x^2}\right).$$

Prove that for nonnegative integers m and n, $\int_{-\infty}^{\infty} A_m(x)A_n(x)dx = \delta(m,n)$, where $\delta(m,n)=1$ if $m=n$, and otherwise $\delta(m,n)=0$.

Proof. *Solution by Tewodros Amdeberhan, Tulane University and Shalosh B. Ekhad, Rutgers Unviversity, NJ, USA.* Convert the differentiation into a contour integral (about a small circle centered at $z=x$) so that

$$A_n(x) = \int_C F_n(z,x)dz, \qquad \text{where} \qquad F_n(z,x) = \frac{(1+x^2)^{n/2}}{(1+z^2)(z-x)^{n+1}}.$$

Apply the Almkvist-Zeilberger algorithm which provides

$$\sqrt{x^2+1}F_{n+2}(z,x) + 2xF_{n+1}(z,x) + \sqrt{x^2+1}F_n(z,x) = \frac{d}{dz}\left(\frac{-(1+z^2)\sqrt{x^2+1}}{n+1}F_n(z,x)\right). \tag{1}$$

Consequently, contour integration on both sides shows $A_n(x)$ to satisfy the three-term recurrence

$$\sqrt{x^2+1}A_{n+2}(x) + 2xA_{n+1}(x) + \sqrt{x^2+1}A_n(x) = 0$$

since the contour integral on the right-hand side of equation (1) is evidently 0. This implies a recurrence for $B(n,m) := \int_{\mathbb{R}} A_n(x)A_m(x)dx$ that is also satisfied by $\delta(m,n)$, and the same initial conditions. Or, make the substitution $y=\frac{x}{\sqrt{x^2+1}}$ to transform $A_n(x)$ into the familiar Chebyshev polynomials of the 2^{nd}-kind $U_n(y)$ and $\int_{\mathbb{R}} A_n(x)A_m(x)dx = \frac{2}{\pi}\int_{-1}^{1} U_n(y)U_m(y)\sqrt{1-y^2}\,dy = \delta(m,n)$. The proof follows. □