Mathematically inspired musical scales

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- Andrew V. Sills (2025) Generalized Carlos scales, Journal of Mathematics and Music 19:3, 243–249, DOI: 10.1080/17459737.2025.2568834

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- Joint work with Robert Schneider, front man for the indie rock band *The Apples in Stereo*, and Assistant Professor of Mathematics, Michigan Technological University.

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- Let us arbitrarily choose one particular pitch as a reference point and call it the *central pitch*. The frequency in Hertz (cycles per second) will be denoted F_0.
- Every other pitch can be written as a positive real multiple of F_0.

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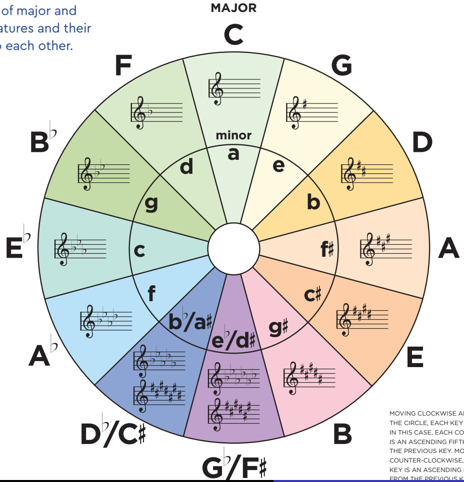
- The difference between 2^7 and 1.5^{12} is called the *Pythagorean comma*, and has to be dealt with when tuning musical instruments.
- Pythagoras pointed out that intervals with frequency ratios of small whole numbers (e.g. 2/1, 3/2, 4/3, 5/4, 6/5, etc.) are pleasing to the ear.

Circle of Fifths

The Circle of Fifths is a graphical representation of major and minor key signatures and their relationships to each other.



MOVING CLOCKWISE AROUND THE CIRCLE, EACH KEY (OR IN THIS CASE, EACH COLOR) IS AN ASCENDING FIFTH FROM THE PREVIOUS KEY. MOVING COUNTER-CLOCKWISE, EACH KEY IS AN ASCENDING FOURTH FROM THE PREVIOUS KEY.

The solution settled on, at least in Western music, is to equally divide the octave into twelve equal intervals:

12-tone equal temperament

The solution settled on, at least in Western music, is to equally divide the octave into twelve equal intervals: the ratio from one note to the next possible lower note is $2^{1/12}$.

Just versus 12ET

$n \downarrow$ $\frac{F_n}{F_0} \rightarrow$	solfège	just	12ET $2^{n/12}$
0	do	1:1 = 1.00000	1.00000
1	di/ra	16:15 = 1.06667	1.05946
2	re	9:8 = 1.12500	1.12246
3	ri/me	6:5 = 1.20000	1.18921
4	mi	5:4 = 1.25000	1.25992
5	fa	4:3 = 1.33333	1.33484
6	fi/se	45:32 = 1.40625	1.41421
7	sol	3:2 = 1.50000	1.49831
8	si/le	8:5 = 1.60000	1.58740
9	la	5:3 = 1.66667	1.68179
10	li/te	9:5 = 1.80000	1.78180
11	ti	15:8 = 1.87500	1.88775
12	do	2:1 = 2.00000	2.00000

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NOTE: Here, a “step” is not meant in terms of the western music theoretic whole/half step nomenclature. Our “step” is a generalization of the half-step/semitone of western music theory.

Noticing that the just scale has unequal steps, we may want to consider dividing a given interval (e.g. the octave) into unequal steps, inspired by functions from mathematics.

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Robert Schneider (2007) created a musical scale which divides the octave (unequally) according to certain values of the base-2 logarithm:

Schneider's Log Scale (SLS)

$n \downarrow$ $\frac{F_n}{F_0} \rightarrow$	solfège	SLS $\log_2(n+4)$	decimal approx
0	do	$\log_2(4)$	2.000000000
1	di/ra	$\log_2(5)$	2.321928094
2	re	$\log_2(6)$	2.584962500
3	ri/me	$\log_2(7)$	2.807354922
4	mi	$\log_2(8)$	3.000000000
5	fa	$\log_2(9)$	3.169925002
6	fi/se	$\log_2(10)$	3.321928095
7	sol	$\log_2(11)$	3.459431619
8	si/le	$\log_2(12)$	3.584962501
9	la	$\log_2(13)$	3.700439717
10	li/te	$\log_2(14)$	3.807354922
11	ti	$\log_2(15)$	3.906890595
12	do	$\log_2(16)$	4.000000000

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$n \downarrow$ $\frac{F_n}{F_0} \rightarrow$	solfège	SLS $\frac{1}{2}\log_2(n+4)$	decimal approx
0	do	$\frac{1}{2}\log_2(4)$	1.000000000
1	di/ra	$\frac{1}{2}\log_2(5)$	1.160964047
2	re	$\frac{1}{2}\log_2(6)$	1.292481250
3	ri/me	$\frac{1}{2}\log_2(7)$	1.403677461
4	mi	$\frac{1}{2}\log_2(8)$	1.500000000
5	fa	$\frac{1}{2}\log_2(9)$	1.584962501
6	fi/se	$\frac{1}{2}\log_2(10)$	1.660964048
7	sol	$\frac{1}{2}\log_2(11)$	1.729715810
8	si/le	$\frac{1}{2}\log_2(12)$	1.792481250
9	la	$\frac{1}{2}\log_2(13)$	1.850219858
10	li/te	$\frac{1}{2}\log_2(14)$	1.903677461
11	ti	$\frac{1}{2}\log_2(15)$	1.953445298
12	do	$\frac{1}{2}\log_2(16)$	2.000000000

Schneider's Log Scale (SLS)

Robert Schneider, *Songs for Other Worlds*, 2022.

https://cloudrecordings.bandcamp.com/album/songs-for-other-worlds-2

A square root scale

$n \downarrow$ $\frac{F_n}{F_0} \rightarrow$	solfège	SQRT $\frac{1}{2}\sqrt{n+4}$	decimal approx
0	do	$\frac{1}{2}\sqrt{4}$	1.000000000
1	di/ra	$\frac{1}{2}\sqrt{5}$	1.118033988
2	re	$\frac{1}{2}\sqrt{6}$	1.224744872
3	ri/me	$\frac{1}{2}\sqrt{7}$	1.322875656
4	mi	$\frac{1}{2}\sqrt{8}$	1.414213562
5	fa	$\frac{1}{2}\sqrt{9}$	1.500000000
6	fi/se	$\frac{1}{2}\sqrt{10}$	1.581138830
7	sol	$\frac{1}{2}\sqrt{11}$	1.658312395
8	si/le	$\frac{1}{2}\sqrt{12}$	1.732050808
9	la	$\frac{1}{2}\sqrt{13}$	1.802775638
10	li/te	$\frac{1}{2}\sqrt{14}$	1.870828694
11	ti	$\frac{1}{2}\sqrt{15}$	1.936491673
12	do	$\frac{1}{2}\sqrt{16}$	2.000000000

An arctangent scale

$n \downarrow$ $\frac{F_n}{F_0} \rightarrow$	solfège	ATN $\frac{2}{\pi}\arctan\left(\frac{n-6}{6}\right)+\frac{3}{2}$	decimal approx
0	do	$\frac{2}{\pi}\arctan\left(-1\right)+\frac{3}{2}$	1.00000
1	di/ra	$\frac{2}{\pi}\arctan\left(-\frac{5}{6}\right)+\frac{3}{2}$	1.05772
2	re	$\frac{2}{\pi}\arctan\left(-\frac{2}{3}\right)+\frac{3}{2}$	1.12567
3	ri/me	$\frac{2}{\pi}\arctan\left(-\frac{1}{2}\right)+\frac{3}{2}$	1.20483
4	mi	$\frac{2}{\pi}\arctan\left(-\frac{1}{3}\right)+\frac{3}{2}$	1.29517
5	fa	$\frac{2}{\pi}\arctan\left(-\frac{1}{6}\right)+\frac{3}{2}$	1.39486
6	fi/se	$\frac{2}{\pi}\arctan\left(0\right)+\frac{3}{2}$	1.50000
7	sol	$\frac{2}{\pi}\arctan\left(\frac{1}{6}\right)+\frac{3}{2}$	1.60514
8	si/le	$\frac{2}{\pi}\arctan\left(\frac{1}{3}\right)+\frac{3}{2}$	1.70483
9	la	$\frac{2}{\pi}\arctan\left(\frac{1}{2}\right)+\frac{3}{2}$	1.79517
10	li/te	$\frac{2}{\pi}\arctan\left(\frac{2}{3}\right)+\frac{3}{2}$	1.87433
11	ti	$\frac{2}{\pi}\arctan\left(\frac{5}{6}\right)+\frac{3}{2}$	1.94228
12	do	$\frac{2}{\pi}\arctan\left(1\right)+\frac{3}{2}$	2.00000

Michigan Tech Mathematics and Music Lab (founded 2022, Robert Schneider): VCV Rack Modules

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To measure small differences in pitch, a *cent* is $1/100$ of a 12ET semitone.

$$\text{cents} = 1200 \log_2(f_2/f_1),$$

where $f_1 < f_2$ are frequencies.

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Carlos used the alpha and beta scales in the album *Beauty in the Beast*.

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Choose positive integers $a < b$ such that a steps approximates a just minor third, b steps approximates a just major third, and $a + b$ steps approximates a perfect fifth.

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Thus for some x,

$$(2^x)^a \approx 6/5 \qquad (2^x)^b \approx 5/4, \text{ and } (2^x)^{a+b} \approx 3/2.$$

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to find the unique minimizer

$$x_0^{(a,b)} = \frac{1}{a^2 + b^2 + (a+b)^2} \log_2 \left[\left(\frac{6}{5}\right)^a \left(\frac{5}{4}\right)^b \left(\frac{3}{2}\right)^{a+b} \right].$$

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Recall that a frequency ratio of $(3/2)^{1/20} \approx 35.098$ cents.
Note, however, that there is no (integer) number of steps that gives a pleasing octave.

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- $(17, 21)$ is also good: 65 steps is very close to a perfect octave.

THANK YOU!