Mechanical proofs of partition identities through atomic relations

Matthew C. Russell

February 5, 2026

Purpose of the talk

- Focus on a technique (atomic relations) I have used in five papers (and counting)
- Discuss computational implementations of the technique
- Sweep other details under the rug

Outline

- **Integer partitions and Rogers–Ramanujan identities — toy example**
- Atomic relations and application to a (mod 10) identity
- Cylindric partitions and applications to infinite families of identities
- Colored partition identities (time permitting)

Integer partitions

A **partition** of n is a way to write n as the sum of positive integers (called **parts**) written in nonincreasing order.

There are five partitions of 4:

$$\begin{aligned} 4 &= 4 & &= 2+1+1 \\ &= 3+1 & &= 1+1+1+1 \\ &= 2+2 \end{aligned}$$

We may also consider generalizations, such as colored partitions or cylindric partitions.

Colored partitions: 5_R is different from 5_B.

Partition identities

Place restrictions on our partitions:

- Only certain types of parts are allowed
- Restrictions on how parts interact with each other

Partition identity: for all n, the number of partitions of n under one set of conditions equals the number of partitions of n under a different set of conditions.

Theorem (First Rogers–Ramanujan identity)

Let n be a nonnegative integer.

- *Let $RR(n)$ be the number of partitions of n where the difference between any two parts is at least 2.*
- *Let $A(n)$ be the number of partitions of n into parts congruent to 1 or $4 \pmod 5$.*

Then, $RR(n) = A(n)$.

Partition identities: generating function versions

Theorem (First Rogers–Ramanujan identity)

Let n be a nonnegative integer.

- *Let $RR(n)$ be the number of partitions of n where the difference between any two parts is at least 2.*
- *Let $A(n)$ be the number of partitions of n into parts congruent to 1 or $4 \pmod 5$.*

Then, $\sum_{n\geq 0} RR(n)q^n = \sum_{n\geq 0} A(n)q^n$.

In fact,
$$\sum_{n\geq 0} RR(n)q^n = \sum_{m\geq 0} \frac{q^{m^2}}{(q;q)_m} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}.$$

q-Pochhammer notation:

$$(a;q)_n = \prod_{0\leq t<n} (1-aq^t), n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

Idea of proof

Want to prove:

$$\sum_{n\geq 0} RR(n)q^n = \sum_{m\geq 0} \frac{q^{m^2}}{(q;q)_m} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}.$$

Second equality follows from Jacobi Triple Product.

For first equality: q is keeping track of the sum of the parts. Let x be another variable that keeps track of the number of parts, and define $RR(n,j)$ to be the number of partitions of n where the difference between any two parts is at least 2 with j parts. Then,

$$\sum_{n,j\geq 0} RR(n,j)\, x^j q^n = \sum_{m\geq 0} \frac{x^m q^{m^2}}{(q;q)_m}.$$

We can prove this by showing both sides satisfy

$$F(x) = F(xq) + xq\, F\left(xq^2\right)$$

(also have to check initial conditions).

Functional equation for first Rogers–Ramanujan identity

If $F(x) = \sum_{n,j \geq 0} RR(n,j)\, x^j q^n$, then:

Consider a partition satisfying the RR conditions: difference between any two parts is at least 2.

Is its smallest part equal to 1?

- If no (all parts ≥ 2), then these partitions are counted by $F(xq)$.
- If yes, "peel off" the smallest part (contributing a factor of xq to the generating function). The leftover partition must have smallest part at least 3. Such partitions are counted by $F\left(xq^2\right)$.

Putting this together:

$$F(x) = F(xq) + xq\, F\left(xq^2\right).$$

Functional equation for first Rogers–Ramanujan identity

If $F(x) = \sum_{m \geq 0} \frac{x^m q^{m^2}}{(q;q)_m}$, then:

$$\begin{aligned} F(x) - F(xq) &= \sum_{m \geq 0} \frac{x^m q^{m^2}}{(q;q)_m} (1 - q^m) \\ &= \sum_{m \geq 0} \frac{x^m q^{m^2}}{(q;q)_{m-1}} \\ &= \sum_{\widehat{m} \geq 0} \frac{x^{\widehat{m}+1} q^{(\widehat{m}+1)^2}}{(q;q)_{\widehat{m}}} \\ &= xq \sum_{\widehat{m} \geq 0} \frac{x^{\widehat{m}} q^{\widehat{m}^2 + 2\widehat{m}}}{(q;q)_{\widehat{m}}} \\ &= xq\, F\left(xq^2\right). \end{aligned}$$

Outline

- Integer partitions and Rogers–Ramanujan identities — toy example
- **Atomic relations and application to a (mod 10) identity**
- Cylindric partitions and applications to infinite families of identities
- Colored partition identities (time permitting)

Main idea of talk

Our main goal will be to show that certain multisums satisfy given functional equations, which can be used to prove identities like this:

$$\sum_{i,j,k,\ell\geq 0} \frac{q^{2i^2+6ij+4ik+2i\ell+6j^2+8jk+4j\ell+3k^2+3k\ell+\ell^2}}{(q^2;q^2)_i\,(q^2;q^2)_j\,(q;q)_k\,(q;q)_\ell}$$
$$= \frac{1}{(q;q^5)_\infty\,(q^4;q^5)_\infty} \cdot \frac{1}{(q^2;q^{10})_\infty\,(q^8;q^{10})_\infty}$$

We want a systematic way to do this that requires as little cleverness as possible.

Solution: **Atomic relations.**

Atomic relations

Consider

$$S_{A,B,C,D}(x,y,q) =$$

$$\sum_{i,j,k,\ell\geq 0} \frac{x^{2j+k+\ell} y^{i+j+k} q^{\binom{2i^2+6ij+4ik+2i\ell+6j^2+8jk+4j\ell+3k^2}{+3k\ell+\ell^2+Ai+Bj+Ck+D\ell}}}{(q^2;q^2)_i (q^2;q^2)_j (q;q)_k (q;q)_\ell}$$

A, B, C, and D are the coefficients of the linear terms in the exponent of q.

We inserted a factor of $x^{2j+k+\ell} y^{i+j+k}$ into our summation (we'll talk later about where this came from).

Recall $\left(q^2;q^2\right)_i = \left(1-q^2\right) \cdots \left(1-q^{2i-2}\right) \left(1-q^{2i}\right)$:
let's multiply the multisum with $\left(1-q^{2i}\right)$.

Atomic relations

Note first that

$$
\begin{aligned}
&S_{A,B,C,D} - S_{A+2,B,C,D} \\
&= \sum_{i,j,k,\ell \geq 0} \frac{x^{2j+k+\ell} y^{i+j+k} q^{\binom{2i^2+6ij+4ik+2i\ell+6j^2+8jk+4j\ell+3k^2}{+3k\ell+\ell^2+Ai+Bj+Ck+D\ell}}}{(q^2;q^2)_i (q^2;q^2)_j (q;q)_k (q;q)_\ell} \\
&\quad - \sum_{i,j,k,\ell \geq 0} \frac{x^{2j+k+\ell} y^{i+j+k} q^{\binom{2i^2+6ij+4ik+2i\ell+6j^2+8jk+4j\ell+3k^2}{+3k\ell+\ell^2+(A+2)i+Bj+Ck+D\ell}}}{(q^2;q^2)_i (q^2;q^2)_j (q;q)_k (q;q)_\ell} \\
&= \sum_{i,j,k,\ell \geq 0} \frac{x^{2j+k+\ell} y^{i+j+k} q^{\binom{2i^2+6ij+4ik+2i\ell+6j^2+8jk+4j\ell+3k^2}{+3k\ell+\ell^2+Ai+Bj+Ck+D\ell}}}{(q^2;q^2)_i (q^2;q^2)_j (q;q)_k (q;q)_\ell} \left(1 - q^{2i}\right).
\end{aligned}
$$

Atomic relations

Now,

$$\sum_{i,j,k,\ell\geq 0} \frac{x^{2j+k+\ell} y^{i+j+k} q^{\left(\substack{2i^2+6ij+4ik+2i\ell+6j^2+8jk+4j\ell+3k^2\\+3k\ell+\ell^2+Ai+Bj+Ck+D\ell}\right)}}{(q^2;q^2)_i\,(q^2;q^2)_j\,(q;q)_k\,(q;q)_\ell}\left(1-q^{2i}\right)$$

$$= \sum_{i,j,k,\ell\geq 0} \frac{x^{2j+k+\ell} y^{i+j+k} q^{\left(\substack{2i^2+6ij+4ik+2i\ell+6j^2+8jk+4j\ell+3k^2\\+3k\ell+\ell^2+Ai+Bj+Ck+D\ell}\right)}}{(q^2;q^2)_{i-1}\,(q^2;q^2)_j\,(q;q)_k\,(q;q)_\ell}$$

$$= \sum_{\widehat{i},j,k,\ell\geq 0} \frac{x^{2j+k+\ell} y^{(\widehat{i}+1)+j+k} q^{\left(\substack{2(\widehat{i}+1)^2+6(\widehat{i}+1)j+4(\widehat{i}+1)k+2(\widehat{i}+1)\ell+6j^2+8jk\\+4j\ell+3k^2+3k\ell+\ell^2+A(\widehat{i}+1)+Bj+Ck+D\ell}\right)}}{(q^2;q^2)_{\widehat{i}}\,(q^2;q^2)_j\,(q;q)_k\,(q;q)_\ell}$$

Here, we reïndexed $i-1=\widehat{i}$, so $i=\widehat{i}+1$.

Atomic relations

Rewrite this:

$$= \sum_{\hat{i},j,k,\ell\geq 0} \frac{x^{2j+k+\ell} y^{(\hat{i}+1)+j+k} q^{\binom{2(\hat{i}+1)^2+6(\hat{i}+1)j+4(\hat{i}+1)k+2(\hat{i}+1)\ell+6j^2+8jk}{+4j\ell+3k^2+3k\ell+\ell^2+A(\hat{i}+1)+Bj+Ck+D\ell}}}{(q^2;q^2)_{\hat{i}}(q^2;q^2)_j(q;q)_k(q;q)_\ell}$$

$$= yq^{A+2} \sum_{\hat{i},j,k,\ell\geq 0} \frac{x^{2j+k+\ell} y^{\hat{i}+j+k} q^{\binom{2\hat{i}^2+4\hat{i}+6(\hat{i}+1)j+4(\hat{i}+1)k+2(\hat{i}+1)\ell+6j^2+8jk}{+4j\ell+3k^2+3k\ell+\ell^2+A\hat{i}+Bj+Ck+D\ell}}}{(q^2;q^2)_{\hat{i}}(q^2;q^2)_j(q;q)_k(q;q)_\ell}$$

$$= yq^{A+2} \sum_{\hat{i},j,k,\ell\geq 0} \frac{x^{2j+k+\ell} y^{\hat{i}+j+k} q^{\binom{2\hat{i}^2+6\hat{i}j+4\hat{i}k+2\hat{i}\ell+6j^2+8jk+4j\ell+3k^2}{+3k\ell+\ell^2+(A+4)\hat{i}+(B+6)j+(C+4)k+(D+2)\ell}}}{(q^2;q^2)_{\hat{i}}(q^2;q^2)_j(q;q)_k(q;q)_\ell}$$

$$= yq^{A+2} S_{A+4,B+6,C+4,D+2}.$$

Atomic relations

Putting this all together: if

$$S_{A,B,C,D}(x,y,q) = \sum_{i,j,k,\ell \geq 0} \frac{x^{2j+k+\ell} y^{i+j+k} q^{\binom{2i^2+6ij+4ik+2i\ell+6j^2+8jk+4j\ell+3k^2}{+3k\ell+\ell^2+Ai+Bj+Ck+D\ell}}}{(q^2;q^2)_i (q^2;q^2)_j (q;q)_k (q;q)_\ell},$$

then

$$S_{A,B,C,D}(x,y,q) - S_{A+2,B,C,D}(x,y,q) = yq^{A+2} S_{A+4,B+6,C+4,D+2}(x,y,q).$$

This is an example of an atomic relation.

Atomic relations

Repeat this for $\left(1-q^{2j}\right)$, $\left(1-q^{k}\right)$, and $\left(1-q^{\ell}\right)$:

$$\begin{aligned}
\mathrm{rel}^1_{A,B,C,D}: &\quad S_{A,B,C,D} - S_{A+2,B,C,D} - yq^{2+A} S_{A+4,B+6,C+4,D+2} &= 0\\
\mathrm{rel}^2_{A,B,C,D}: &\quad S_{A,B,C,D} - S_{A,B+2,C,D} - x^2yq^{6+B} S_{A+6,B+12,C+8,D+4} &= 0\\
\mathrm{rel}^3_{A,B,C,D}: &\quad S_{A,B,C,D} - S_{A,B,C+1,D} - xyq^{3+C} S_{A+4,B+8,C+6,D+3} &= 0\\
\mathrm{rel}^4_{A,B,C,D}: &\quad S_{A,B,C,D} - S_{A,B,C,D+1} - xq^{1+D} S_{A+2,B+4,C+3,D+2} &= 0
\end{aligned}$$

These are the atomic relations for

$$S_{A,B,C,D} = \sum_{i,j,k,\ell \geq 0} \frac{x^{2j+k+\ell} y^{i+j+k} q^{\left(\substack{2i^2+6ij+4ik+2i\ell+6j^2+8jk+4j\ell+3k^2 \\ +3k\ell+\ell^2+Ai+Bj+Ck+D\ell}\right)}}{(q^2;q^2)_i (q^2;q^2)_j (q;q)_k (q;q)_\ell}.$$

We also have the following shifts of $S_{A,B,C,D}$:

$$\begin{aligned}
S_{A,B,C,D}(xq,y,q) &= S_{A,B+2,C+1,D+1}(x,y,q)\\
S_{A,B,C,D}(x,yq,q) &= S_{A+1,B+1,C+1,D}(x,y,q).
\end{aligned}$$

Applying atomic relations

Recall we want to prove this:

$$\sum_{i,j,k,\ell\geq 0} \frac{q^{2i^2+6ij+4ik+2i\ell+6j^2+8jk+4j\ell+3k^2+3k\ell+\ell^2}}{(q^2;q^2)_i\,(q^2;q^2)_j\,(q;q)_k\,(q;q)_\ell}$$
$$= \frac{1}{(q;q^5)_\infty\,(q^4;q^5)_\infty} \cdot \frac{1}{(q^2;q^{10})_\infty\,(q^8;q^{10})_\infty}$$

Instead, we will prove:

$$S_{0,0,0,0}(x,y,q) = \left(\sum_{i\geq 0} \frac{x^i q^{i^2}}{(q;q)_i}\right)\left(\sum_{j\geq 0} \frac{y^j q^{2j^2}}{(q^2;q^2)_j}\right).$$

Specializing $x, y \mapsto 1$ recovers the desired result (after using the first Rogers–Ramanujan identity).

Applying atomic relations

$$\left(\sum_{i\geq 0} \frac{x^i q^{i^2}}{(q;q)_i}\right)\left(\sum_{j\geq 0} \frac{y^j q^{2j^2}}{(q^2;q^2)_j}\right)$$

satisfies the functional equation

$$F(x,y,q) = F(xq,y,q) + xqF\left(xq^2,y,q\right).$$

Need to show that $S_{0,0,0,0}(x,y,q)$ also satisfies this functional equations. Use the shift

$$S_{A,B,C,D}(xq,y,q) = S_{A,B+2,C+1,D+1}(x,y,q)$$

to produce

$$S_{0,0,0,0} - S_{0,2,1,1} - xqS_{0,4,2,2} = 0.$$

Can we write this as a linear combination of atomic relations?

Applying atomic relations

Yes!

$$
\begin{aligned}
& -\operatorname{rel}^1_{-2,0,0,0} + \operatorname{rel}^1_{-2,0,0,1} - xq \cdot \operatorname{rel}^1_{0,4,2,2} + xq \cdot \operatorname{rel}^1_{0,4,3,2} + \operatorname{rel}^2_{0,0,0,1} \\
& + \operatorname{rel}^3_{0,2,0,1} - xq \cdot \operatorname{rel}^3_{2,4,2,2} + \operatorname{rel}^4_{-2,0,0,0} - y \cdot \operatorname{rel}^4_{2,6,4,2} \\
& = -\left(S_{-2,0,0,0} - S_{0,0,0,0} - yS_{2,6,4,2}\right) + \left(S_{-2,0,0,1} - S_{0,0,0,1} - yS_{2,6,4,3}\right) \\
& -xq\left(S_{0,4,2,2} - S_{2,4,2,2} - yq^2 S_{4,10,6,4}\right) + xq\left(S_{0,4,3,2} - S_{2,4,3,2} - yq^2 S_{4,10,7,4}\right) \\
& +\left(S_{0,0,0,1} - S_{0,2,0,1} - x^2yq^6 S_{6,12,8,5}\right) + \left(S_{0,2,0,1} - S_{0,2,1,1} - xyq^3 S_{4,10,6,4}\right) \\
& -xq\left(S_{2,4,2,2} - S_{2,4,3,2} - xyq^5 S_{6,12,8,5}\right) + \left(S_{-2,0,0,0} - S_{-2,0,0,1} - xqS_{0,4,3,2}\right) \\
& - y\left(S_{2,6,4,2} - S_{2,6,4,3} - xq^3 S_{4,10,7,4}\right)
\end{aligned}
$$

Applying atomic relations

Yes!

$$
\begin{aligned}
&-\mathrm{rel}^1_{-2,0,0,0}+\mathrm{rel}^1_{-2,0,0,1}-xq\cdot\mathrm{rel}^1_{0,4,2,2}+xq\cdot\mathrm{rel}^1_{0,4,3,2}+\mathrm{rel}^2_{0,0,0,1}\\
&+\mathrm{rel}^3_{0,2,0,1}-xq\cdot\mathrm{rel}^3_{2,4,2,2}+\mathrm{rel}^4_{-2,0,0,0}-y\cdot\mathrm{rel}^4_{2,6,4,2}\\
&=-\left(S_{-2,0,0,0}-S_{0,0,0,0}-yS_{2,6,4,2}\right)+\left(S_{-2,0,0,1}-S_{0,0,0,1}-yS_{2,6,4,3}\right)\\
&-xq\left(S_{0,4,2,2}-S_{2,4,2,2}-yq^2S_{4,10,6,4}\right)+xq\left(S_{0,4,3,2}-S_{2,4,3,2}-yq^2S_{4,10,7,4}\right)\\
&+\left(S_{0,0,0,1}-S_{0,2,0,1}-x^2yq^6S_{6,12,8,5}\right)+\left(S_{0,2,0,1}-S_{0,2,1,1}-xyq^3S_{4,10,6,4}\right)\\
&-xq\left(S_{2,4,2,2}-S_{2,4,3,2}-xyq^5S_{6,12,8,5}\right)+\left(S_{-2,0,0,0}-S_{-2,0,0,1}-xqS_{0,4,3,2}\right)\\
&-y\left(S_{2,6,4,2}-S_{2,6,4,3}-xq^3S_{4,10,7,4}\right)\\
&=S_{0,0,0,0}-S_{0,2,1,1}-xqS_{0,4,2,2}\qquad=0.
\end{aligned}
$$

Applying atomic relations

$$
\begin{aligned}
&-\mathrm{rel}^1_{-2,0,0,0} + \mathrm{rel}^1_{-2,0,0,1} - xq \cdot \mathrm{rel}^1_{0,4,2,2} + xq \cdot \mathrm{rel}^1_{0,4,3,2} + \mathrm{rel}^2_{0,0,0,1} \\
&+ \mathrm{rel}^3_{0,2,0,1} - xq \cdot \mathrm{rel}^3_{2,4,2,2} + \mathrm{rel}^4_{-2,0,0,0} - y \cdot \mathrm{rel}^4_{2,6,4,2} \\
&= S_{0,0,0,0} - S_{0,2,1,1} - xqS_{0,4,2,2} \\
&= 0.
\end{aligned}
$$

After checking initial conditions (in this case, requires some inductive arguments), conclude that

$$
S_{0,0,0,0}(x,y,q) = \left(\sum_{i \geq 0} \frac{x^i q^{i^2}}{(q;q)_i} \right) \left(\sum_{j \geq 0} \frac{y^j q^{2j^2}}{(q^2;q^2)_j} \right).
$$

Recapitulation

Goal: prove multisum = product identity.

- Insert in auxiliary variables (x, y, etc.) into both sides
- Deduce functional equation for product side
- Write down atomic relations for multisums
- Show that the multisums satisfy the functional equation, by writing as a linear combination of atomic relations
- Check initial conditions

Inserting auxiliary variables

- Insert in one auxiliary variable for each summation variable on the left
- Rewrite the right side in a way that is amenable to the introduction of auxiliary variables
- Expand both as Taylor series and compare coefficients.

No guarantee this will work!

$$\sum_{i,j,k,\ell\geq 0} \frac{t^i u^j v^k w^\ell q^{2i^2+6ij+4ik+2i\ell+6j^2+8jk+4j\ell+3k^2+3k\ell+\ell^2}}{(q^2;q^2)_i (q^2;q^2)_j (q;q)_k (q;q)_\ell}$$
$$= \left(\sum_{i\geq 0} \frac{x^i q^{i^2}}{(q;q)_i}\right) \left(\sum_{j\geq 0} \frac{y^j q^{2j^2}}{(q^2;q^2)_j}\right)$$

Auxiliary variables

$$\sum_{i,j,k,\ell\geq 0} \frac{t^i u^j v^k w^\ell q^{2i^2+6ij+4ik+2i\ell+6j^2+8jk+4j\ell+3k^2+3k\ell+\ell^2}}{(q^2;q^2)_i\,(q^2;q^2)_j\,(q;q)_k\,(q;q)_\ell}$$

$$= \left(\sum_{i\geq 0} \frac{x^i q^{i^2}}{(q;q)_i}\right)\left(\sum_{j\geq 0} \frac{y^j q^{2j^2}}{(q^2;q^2)_j}\right)$$

$$\begin{aligned} &1 + wq + (t+w)q^2 + (v+w)q^3 + (w^2+t+v+w)q^4 \\ &\quad + (tw+w^2+v+w)q^5 + (tw+2w^2+t+u+v+w)q^6 + \cdots \\ =&1 + xq + (y+x)q^2 + (xy+x)q^3 + (x^2+xy+x+y)q^4 \\ &\quad + (x^2+2xy+x)q^5 + (y+2xy+x^2y+2x^2+x)q^6 + \cdots \end{aligned}$$

Solution: $w = x,\ t = y,\ v = xy,\ u = x^2y$

Finding linear combination of atomic relations: with Maple!

- Create a generic linear combination of $\mathrm{rel}^i_{A,B,C,D}$ $(1 \leq i \leq 4)$ with coefficients $u^i_{A,B,C,D}$ for, say, $a_0 \leq A \leq a_1$, $b_0 \leq B \leq b_1$, $c_0 \leq C \leq c_1$, $d_0 \leq D \leq d_1$
- Write out each $\mathrm{rel}^i_{A,B,C,D}$ in terms of $S_{A,B,C,D}$ ($\mathrm{rel}^1_{A,B,C,D}$: $S_{A,B,C,D} - S_{A+2,B,C,D} - yq^{2+A} S_{A+4,B+6,C+4,D+2}$, etc.)
- Subtract the desired functional equation $(S_{0,0,0,0} - S_{0,2,1,1} - xqS_{0,4,2,2})$ from the generic linear combination
- Linear algebra problem: need every coefficient of $S_{A,B,C,D}$ to equal zero; solve the resulting system of linear equations

Finding linear combination of atomic relations: with Maple!

- Coefficients are (Laurent) polynomials in formal variables q, x, y: try setting these to small natural numbers ($q = 2$, $x = 3$, $y = 5$)
- Set free variables in the resulting solution to zero
- Solve again setting, say, $q = \pi$, $x = \gamma$, $y = \exp(1)$ (easy to convert back)
- Check that the linear combination of the rel^is you obtain does give you the correct functional equation

References

- MCR. On a pair of three-colored (mod 10) partition identities. `arXiv:2509.07169`
- S. Kanade and MCR. Tight cylindric partitions. `arXiv:2508.15113`
- MCR. Companions to the Andrews-Gordon and Andrews-Bressoud identities, and recent conjectures of Capparelli, Meurman, Primc, and Primc. `arXiv:2306.16251`
- K. Baker, S. Kanade, MCR, and and C. Sadowski. Principal subspaces of basic modules for twisted affine Lie algebras, q-series multisums, and Nandi's identities. *Algebraic Combinatorics* (2023) `arXiv:2208.14581`
- S. Kanade and MCR. Completing the A_2 Andrews-Schilling-Warnaar identities. *International Mathematics Research Notices* (2023) `arXiv:2203.05690`

Baker, Kanade, MCR, and Sadowski: "Principal subspaces of basic modules..."

- We proved four separate quadruple-sum-equals-(mod 10)-product identities (we already saw one)
- In addition, we found new quadruple-sum representations of the sum sides of Nandi's (mod 14) partition identities (now theorems due to Takigiku and Tsuchioka).

Outline

- Integer partitions and Rogers–Ramanujan identities — toy example
- Atomic relations and application to a (mod 10) identity
- **Cylindric partitions and applications to infinite families of identities**
- Colored partition identities (time permitting)

Kanade and MCR: "Tight cylindric partitions" and MCR: "...Capparelli, Meurman, Primc, and Primc."

We proved infinite families of multisum identities related to cylindric partitions and colored partitions, where computers were used to guide discovery of the necessary linear combinations of atomic relations, but final results all proved by hand.

Cylindric partitions

Introduced in 1997 by Gessel and Krattenthaler.

Take a composition $(c_1, \ldots, c_r)$ of ℓ, where $c_1, \ldots, c_r$ are non-negative integers and $\ell = c_1 + \cdots + c_r$.

Consider a sequence $\Lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ where each $\lambda^{(j)}$ is a partition $\lambda^{(j)} = \lambda^{(j)}_1 + \lambda^{(j)}_2 + \cdots$ arranged in a weakly descending order.

We say that Λ is a cylindric partition with profile c if for all i and j,

$$\lambda^{(i)}_j \geq \lambda^{(i+1)}_{j+c_{(i+1)}} \quad \text{and} \quad \lambda^{(r)}_j \geq \lambda^{(1)}_{j+c_1}.$$

Example

Profile $(2,1)$:

$$\begin{aligned}\lambda^{(1)} &\to \\ \lambda^{(2)} &\to\end{aligned}$$

Example

Profile $(2,1)$:

$$\begin{array}{llllll} \lambda^{(1)} \rightarrow & & 6 & 3 & 2 & 2 \\ \lambda^{(2)} \rightarrow & 5 & 4 & 1 & & \end{array}$$

Example

Profile $(2,1)$:

$$\begin{array}{lccccccc} \lambda^{(2)} \rightarrow & & & & 5 & 4 & 1 \\ \lambda^{(1)} \rightarrow & & 6 & 3 & 2 & 2 & \\ \lambda^{(2)} \rightarrow & 5 & 4 & 1 & & & \end{array}$$

Tight cylindric partitions

Cylindric partitions in which no positive part appears in every row.

Tight:

$$\begin{array}{lccccccc} \color{gray}{\lambda^{(2)} \rightarrow} & & & & \color{gray}{5} & \color{gray}{4} & \color{gray}{1} \\ \lambda^{(1)} \rightarrow & & 6 & 3 & 2 & 2 & \\ \lambda^{(2)} \rightarrow & 5 & 4 & 1 & & & \end{array}$$

Not tight:

$$\begin{array}{lccccccc} \color{gray}{\lambda^{(2)} \rightarrow} & & & & \color{gray}{5} & \color{gray}{3} & \color{gray}{1} \\ \lambda^{(1)} \rightarrow & & 6 & 3 & 2 & 2 & \\ \lambda^{(2)} \rightarrow & 5 & 3 & 1 & & & \end{array}$$

"Tight": borrowed from tight abaci.

Generating functions

$T_c(z,q) = T_{(c_1,c_2,\ldots,c_r)}(z,q)$: generating function of tight cylindric partitions of profile c.

$$T_c(z,q) = \sum_{\Lambda \in \mathcal{T}_c} z^{\max(\Lambda)} q^{\mathrm{wt}(\Lambda)},$$

$\mathcal{T}_c$ is set of tight cylindric partitions of profile c.

Theorem

For a profile $c = (c_1, c_2)$, we have that:

$$T_{(b,\ell-b)}(1,q) = \frac{(-q;q)_\infty \left(q^{b+1}, q^{\ell-b+1}, q^{\ell+2}; q^{\ell+2}\right)_\infty}{(q;q)_\infty}$$

(Also get nice infinite products for r-rowed cylindric partitions and tight cylindric partitions.)

Diamond relations for two-rowed tight cylindric partitions

Proposition

We have for each $\ell \geq 1$, $1 \leq i \leq \ell - 1$*:*

$$T_{(\ell-i,i)}(z) + \frac{zq^2}{1-zq^2} T_{(\ell-i,i)}(zq^2) = \frac{T_{(\ell-i+1,i-1)}(zq)}{1-zq} + \frac{T_{(\ell-i-1,i+1)}(zq)}{1-zq} - \frac{T_{(\ell-i,i)}(zq^2)}{1-zq^2}$$

and:

$$T_{(\ell,0)}(z) + \frac{zq^2}{1-zq^2} T_{(\ell,0)}(zq^2) = \frac{T_{(\ell-1,1)}(zq)}{1-zq}$$

$$T_{(0,\ell)}(z) + \frac{zq^2}{1-zq^2} T_{(0,\ell)}(zq^2) = \frac{T_{(1,\ell-1)}(zq)}{1-zq}$$

(We also have functional equations for r-rowed tight cylindric partitions.)

Example: $r = 2$, $\ell = 6$

$$T_{(6,0)}(z) = \frac{T_{(5,1)}(zq)}{1 - zq} - \frac{zq^2}{1 - zq^2} T_{(6,0)}(zq^2)$$

$$T_{(5,1)}(z) = \frac{T_{(4,2)}(zq)}{1 - zq} + \frac{T_{(6,0)}(zq)}{1 - zq} - \frac{\left(1 - zq^2\right) T_{(5,1)}(zq^2)}{1 - zq^2}$$

$$T_{(4,2)}(z) = \frac{T_{(3,3)}(zq)}{1 - zq} + \frac{T_{(5,1)}(zq)}{1 - zq} - \frac{\left(1 - zq^2\right) T_{(4,2)}(zq^2)}{1 - zq^2}$$

$$T_{(3,3)}(z) = \frac{T_{(2,4)}(zq)}{1 - zq} + \frac{T_{(4,2)}(zq)}{1 - zq} - \frac{\left(1 - zq^2\right) T_{(3,3)}(zq^2)}{1 - zq^2}$$

$$T_{(2,4)}(z) = \frac{T_{(1,5)}(zq)}{1 - zq} + \frac{T_{(3,3)}(zq)}{1 - zq} - \frac{\left(1 - zq^2\right) T_{(2,4)}(zq^2)}{1 - zq^2}$$

$$T_{(1,5)}(z) = \frac{T_{(0,6)}(zq)}{1 - zq} + \frac{T_{(2,4)}(zq)}{1 - zq} - \frac{\left(1 - zq^2\right) T_{(1,5)}(zq^2)}{1 - zq^2}$$

$$T_{(0,6)}(z) = \frac{T_{(1,5)}(zq)}{1 - zq} - \frac{zq^2}{1 - zq^2} T_{(0,6)}(zq^2)$$

Diamond relations

$$\begin{array}{ccccc}
T_{(6,0)}(z) & T_{(6,0)}(zq) & T_{(6,0)}(zq^2) & T_{(6,0)}(zq^3) & \cdots \\
T_{(5,1)}(z) & T_{(5,1)}(zq) & T_{(5,1)}(zq^2) & T_{(5,1)}(zq^3) & \cdots \\
T_{(4,2)}(z) & T_{(4,2)}(zq) & T_{(4,2)}(zq^2) & T_{(4,2)}(zq^3) & \cdots \\
T_{(3,3)}(z) & T_{(3,3)}(zq) & T_{(3,3)}(zq^2) & T_{(3,3)}(zq^3) & \cdots \\
T_{(2,4)}(z) & T_{(2,4)}(zq) & T_{(2,4)}(zq^2) & T_{(2,4)}(zq^3) & \cdots \\
T_{(1,5)}(z) & T_{(1,5)}(zq) & T_{(1,5)}(zq^2) & T_{(1,5)}(zq^3) & \cdots \\
T_{(0,6)}(z) & T_{(0,6)}(zq) & T_{(0,6)}(zq^2) & T_{(0,6)}(zq^3) & \cdots
\end{array}$$

Diamond relations

$$\begin{array}{ccccc}
T_{(6,0)}(z) & T_{(6,0)}(zq) & T_{(6,0)}(zq^2) & T_{(6,0)}(zq^3) & \cdots \\
T_{(5,1)}(z) & T_{(5,1)}(zq) & T_{(5,1)}(zq^2) & T_{(5,1)}(zq^3) & \cdots \\
T_{(4,2)}(z) & T_{(4,2)}(zq) & T_{(4,2)}(zq^2) & T_{(4,2)}(zq^3) & \cdots \\
T_{(3,3)}(z) & T_{(3,3)}(zq) & T_{(3,3)}(zq^2) & T_{(3,3)}(zq^3) & \cdots \\
T_{(2,4)}(z) & T_{(2,4)}(zq) & T_{(2,4)}(zq^2) & T_{(2,4)}(zq^3) & \cdots \\
T_{(1,5)}(z) & T_{(1,5)}(zq) & T_{(1,5)}(zq^2) & T_{(1,5)}(zq^3) & \cdots \\
T_{(0,6)}(z) & T_{(0,6)}(zq) & T_{(0,6)}(zq^2) & T_{(0,6)}(zq^3) & \cdots
\end{array}$$

$$T_{(4,2)}(z) = \frac{T_{(3,3)}(zq)}{1 - zq} + \frac{T_{(5,1)}(zq)}{1 - zq} - \frac{\left(1 - zq^2\right) T_{(4,2)}(zq^2)}{1 - zq^2}$$

Diamond relations

$$\begin{array}{ccccc}
T_{(6,0)}(z) & T_{(6,0)}(zq) & T_{(6,0)}(zq^2) & T_{(6,0)}(zq^3) & \cdots \\
T_{(5,1)}(z) & T_{(5,1)}(zq) & T_{(5,1)}(zq^2) & T_{(5,1)}(zq^3) & \cdots \\
T_{(4,2)}(z) & T_{(4,2)}(zq) & T_{(4,2)}(zq^2) & T_{(4,2)}(zq^3) & \cdots \\
T_{(3,3)}(z) & T_{(3,3)}(zq) & T_{(3,3)}(zq^2) & T_{(3,3)}(zq^3) & \cdots \\
T_{(2,4)}(z) & T_{(2,4)}(zq) & T_{(2,4)}(zq^2) & T_{(2,4)}(zq^3) & \cdots \\
T_{(1,5)}(z) & T_{(1,5)}(zq) & T_{(1,5)}(zq^2) & T_{(1,5)}(zq^3) & \cdots \\
T_{(0,6)}(z) & T_{(0,6)}(zq) & T_{(0,6)}(zq^2) & T_{(0,6)}(zq^3) & \cdots
\end{array}$$

$$T_{(4,2)}(z) = \frac{T_{(3,3)}(zq)}{1-zq} + \frac{T_{(5,1)}(zq)}{1-zq} - \frac{\left(1-zq^2\right)\, T_{(4,2)}(zq^2)}{1-zq^2}$$

Example: $r = 2$, $\ell = 6$

Use symmetry: $T_{(\ell-b,b)}(z,q) = T_{(b,\ell-b)}(z,q)$.

$$T_{(6,0)}(z) = \frac{T_{(5,1)}(zq)}{1-zq} - \frac{zq^2}{1-zq^2} T_{(6,0)}(zq^2)$$

$$T_{(5,1)}(z) = \frac{T_{(4,2)}(zq)}{1-zq} + \frac{T_{(6,0)}(zq)}{1-zq} - \frac{\left(1-zq^2\right) T_{(5,1)}(zq^2)}{1-zq^2}$$

$$T_{(4,2)}(z) = \frac{T_{(3,3)}(zq)}{1-zq} + \frac{T_{(5,1)}(zq)}{1-zq} - \frac{\left(1-zq^2\right) T_{(4,2)}(zq^2)}{1-zq^2}$$

$$T_{(3,3)}(z) = \frac{2T_{(4,2)}(zq)}{1-zq} - \frac{\left(1-zq^2\right) T_{(3,3)}(zq^2)}{1-zq^2}$$

Bivariate generating functions of two-rowed tight cylindric partitions

Theorem

Fix $\ell \geq 1$. Let $0 \leq i \leq \lfloor \frac{\ell}{2} \rfloor$. We have:

$$T_{(\ell-i,i)}(z,q) = T_{(i,\ell-i)}(z,q)$$
$$= \sum_{n_1,\cdots,n_\ell \geq 0} \frac{z^{N_1} q^{\frac{1}{2}(N_1^2+\cdots N_\ell^2)+\frac{1}{2}(n_1+n_3+\cdots n_{2i-1})+\frac{1}{2}(N_{2i+1}+N_{2i+2}+\cdots+N_\ell)}(q;q)_{N_1}}{(zq;q)_{N_1}(q;q)_{n_1}\cdots(q;q)_{n_\ell}}$$

where, as usual, $N_j = n_j + n_{j+1} + \cdots + n_\ell$.

Example: $\ell = 6$

$$T_{(6,0)}(z,q) = \sum_{\substack{n_1,n_2,n_3,\\ n_4,n_5,n_6 \geq 0}} \frac{z^{N_1} q^{\frac{1}{2}(N_1^2+\cdots+N_6^2)+\frac{1}{2}(n_1+2n_2+3n_3+4n_4+5n_5+6n_6)} (q;q)_{N_1}}{(zq;q)_{N_1} (q;q)_{n_1} \cdots (q;q)_{n_6}}$$

$$T_{(5,1)}(z,q) = \sum_{\substack{n_1,n_2,n_3,\\ n_4,n_5,n_6 \geq 0}} \frac{z^{N_1} q^{\frac{1}{2}(N_1^2+\cdots+N_6^2)+\frac{1}{2}(n_1+0n_2+n_3+2n_4+3n_5+4n_6)} (q;q)_{N_1}}{(zq;q)_{N_1} (q;q)_{n_1} \cdots (q;q)_{n_6}}$$

$$T_{(4,2)}(z,q) = \sum_{\substack{n_1,n_2,n_3,\\ n_4,n_5,n_6 \geq 0}} \frac{z^{N_1} q^{\frac{1}{2}(N_1^2+\cdots+N_6^2)+\frac{1}{2}(n_1+0n_2+n_3+0n_4+n_5+2n_6)} (q;q)_{N_1}}{(zq;q)_{N_1} (q;q)_{n_1} \cdots (q;q)_{n_6}}$$

$$T_{(3,3)}(z,q) = \sum_{\substack{n_1,n_2,n_3,\\ n_4,n_5,n_6 \geq 0}} \frac{z^{N_1} q^{\frac{1}{2}(N_1^2+\cdots+N_6^2)+\frac{1}{2}(n_1+0n_2+n_3+0n_4+n_5+0n_6)} (q;q)_{N_1}}{(zq;q)_{N_1} (q;q)_{n_1} \cdots (q;q)_{n_6}}$$

Proof sketch

Show

$$T_{(\ell-i,i)}(z,q) = \sum_{n_1,\cdots,n_\ell\geq 0} \frac{z^{N_1} q^{\frac{1}{2}(N_1^2+\cdots N_\ell^2)+\frac{1}{2}(n_1+n_3+\cdots n_{2i-1})+\frac{1}{2}(N_{2i+1}+N_{2i+2}+\cdots+N_\ell)}(q;q)_{N_1}}{(zq;q)_{N_1}(q;q)_{n_1}\cdots(q;q)_{n_\ell}}$$

satisfies

$$T_{(\ell-i,i)}(z) = \frac{T_{(\ell-i+1,i-1)}(zq)}{1-zq} + \frac{T_{(\ell-i-1,i+1)}(zq)}{1-zq} - \frac{(1+zq^2)\,T_{(\ell-i,i)}(zq^2)}{1-zq^2}$$

$$T_{(\ell,0)}(z) + \frac{zq^2}{1-zq^2}T_{(\ell,0)}(zq^2) = \frac{T_{(\ell-1,1)}(zq)}{1-zq}$$

$$T_{(0,\ell)}(z) + \frac{zq^2}{1-zq^2}T_{(0,\ell)}(zq^2) = \frac{T_{(1,\ell-1)}(zq)}{1-zq}$$

Proof sketch

Write $\mathbf{v} = (v_1, \cdots, v_\ell)$. Define

$$S(t;\mathbf{v};z,q) = \sum_{n_1,\dots,n_\ell \geq 0} \frac{z^{N_1} q^{\left(N_1^2+N_2^2+\cdots+N_\ell^2+N_1+N_2+\cdots+N_\ell\right)/2+\mathbf{v}\cdot\vec{n}} (q;q)_{N_1}}{(zq;q)_{N_1+t}(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_\ell}}.$$

Also:

$$\begin{aligned} e_j &= [0\ 0\ \ \cdots\ \ 0\ 1\ \ 0\ \ \cdots\ \ 0] \\ \delta_j &= e_j + e_{j+1} + \cdots + e_\ell, \\ \Delta_i &= \delta_1 + \delta_2 + \cdots + \delta_i, \\ \eta_i &= \delta_2 + \delta_4 + \cdots + \delta_{2i}. \end{aligned}$$

If $j > \ell$, then $e_j = \delta_j = 0$. Note that $e_j = \delta_j - \delta_{j+1}$.

Restating main theorem

By symmetry, $T_{(\ell-b,b)}(z,q) = T_{(b,\ell-b)}(z,q)$. Thus, it suffices to show that the appropriate set of multisums satisfy half of the "diamond relations". Writing $R_b(z,q) = T_{(\ell-b,b)}(z,q)$ for $0 \leq b \leq \lfloor \ell/2 \rfloor$, we restate:

Restating main theorem

Theorem

ℓ even: A solution to

$$R_0(z) = \frac{R_1(zq)}{1-zq} - \frac{zq^2}{1-zq^2} R_0(zq^2)$$

$$R_i(z) = \frac{R_{i-1}(zq)}{1-zq} + \frac{R_{i+1}(zq)}{1-zq} - \frac{\left(1+zq^2\right) R_i(zq^2)}{1-zq^2},\ 1 \le i < \frac{\ell-1}{2}$$

$$R_{\ell/2}(z) = \frac{2R_{\ell/2-1}(zq)}{1-zq} - \frac{\left(1+zq^2\right) R_{\ell/2}(zq^2)}{1-zq^2}$$

with initial conditions $R_i(0,q) = R_i(z,0) = 1$, is given by

$$R_i(z,q) = S(0; -\eta_i; z, q)$$

for $0 \le i \le \ell/2$.

Restating main theorem

Theorem

ℓ odd: A solution to

$$R_0(z) = \frac{R_1(zq)}{1-zq} - \frac{zq^2}{1-zq^2} R_0(zq^2)$$

$$R_i(z) = \frac{R_{i-1}(zq)}{1-zq} + \frac{R_{i+1}(zq)}{1-zq} - \frac{\left(1+zq^2\right) R_i(zq^2)}{1-zq^2}, \; 1 \le i < \frac{\ell-1}{2}$$

$$R_{\frac{\ell-1}{2}}(z) = \frac{R_{(\ell-3)/2}(zq)}{1-zq} + \frac{R_{(\ell-1)/2}(zq)}{1-zq} - \frac{\left(1+zq^2\right) R_{(\ell-1)/2}(zq^2)}{1-zq^2}$$

with initial conditions $R_i(0,q) = R_i(z,0) = 1$, is given by

$$R_i(z,q) = S(0; -\eta_i; z, q)$$

for $0 \le i \le \lfloor \ell/2 \rfloor$.

Translating into S notation

Recall

$$S(t;\mathbf{v};z,q) = \sum_{n_1,\ldots,n_\ell \geq 0} \frac{z^{N_1} q^{\left(N_1^2+N_2^2+\cdots+N_\ell^2+N_1+N_2+\cdots+N_\ell\right)/2+\mathbf{v}\cdot\vec{n}} (q;q)_{N_1}}{(zq;q)_{N_1+t}(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_\ell}}.$$

Observe that

$$\frac{S(0;\mathbf{v};zq)}{1-zq} = S(1;\mathbf{v}+\delta_1;z)$$

$$\frac{S\left(0;\mathbf{v};zq^2\right)}{(1-zq)\left(1-zq^2\right)} = S(2;\mathbf{v}+2\delta_1;z)$$

Translating into S notation

We obtain:

$$S(0;\vec{0}) + zq^2(1-zq)S(2;2\delta_1) = S(1;e_1)$$

$$S(0;-\eta_i) + (1-zq)(1+zq^2)S(2;2\delta_1-\eta_i)$$

$$= S(1;\delta_1-\eta_{i-1}) + S(1;\delta_1-\eta_{i+1}), \quad 1 \le i < \frac{\ell-1}{2}$$

$$S(0;-\eta_{\ell/2}) + (1-zq)(1+zq^2)S(2;2\delta_1-\eta_{\ell/2})$$

$$= 2S(1;\delta_1-\eta_{\ell/2-1}), \quad \text{if } \ell \text{ is even}$$

$$S(0;-\eta_{\frac{\ell-1}{2}}) + (1-zq)(1+zq^2)S(2;2\delta_1-\eta_{\frac{\ell-1}{2}})$$

$$= S(1;\delta_1-\eta_{\frac{\ell-3}{2}}) + S(1;\delta_1-\eta_{\frac{\ell-1}{2}}), \quad \text{if } \ell \text{ is odd}$$

Atomic relations

$$\mathrm{rel}_0(t;\mathbf{v}) : S(t;\mathbf{v}) - S(t+1;\mathbf{v}) + zq^{t+1}S(t+1;\mathbf{v}+\delta_1) = 0$$

Subtract the corresponding summands of $S(t;\mathbf{v})$ and $S(t+1;\mathbf{v})$.

For $1 \leq j \leq \ell$:

$$\begin{aligned}\mathrm{rel}_j(t;\mathbf{v}) : S(t;\mathbf{v}) - S(t;\mathbf{v}+e_j) - zq^{v_j+j}S(t+1;\mathbf{v}+\Delta_j) \\ + zq^{v_j+j+1}S(t+1;\mathbf{v}+\delta_1+\Delta_j) = 0.\end{aligned}$$

Multiply each summand of $S(t;\mathbf{v})$ by $(1-q^{n_j})$ (summands corresponding to $n_j = 0$ vanish) and shift summation index $n_j \mapsto n_j + 1$.

Goal: write equations on previous slide as linear combinations of atomic relations.

$i = 0$ equation

$$\mathrm{rel}_0(0;\vec{0}) + \mathrm{rel}_1(1;\vec{0}) - zq\,\mathrm{rel}_0(1;\delta_1)$$

$i = 0$ equation

$$
\begin{aligned}
&\mathrm{rel}_0(0;\vec{0}) + \mathrm{rel}_1(1;\vec{0}) - zq\,\mathrm{rel}_0(1;\delta_1) \\
&= S(0;\vec{0}) - S(1;\vec{0}) + zq\,S(1;\delta_1) \\
&+ S(1;\vec{0}) - S(1;e_1) - zq\,S(2;\delta_1) + zq^2 S(2;2\delta_1) \\
&- zq\left(S(1;\delta_1) - S(2;\delta_1) + zq^2\,S(2;2\delta_1)\right)
\end{aligned}
$$

$i = 0$ equation

$$\mathrm{rel}_0(0;\vec{0}) + \mathrm{rel}_1(1;\vec{0}) - zq\,\mathrm{rel}_0(1;\delta_1)$$
$$= S(0;\vec{0}) - S(1;\vec{0}) + zq\,S(1;\delta_1)$$
$$+ S(1;\vec{0}) - S(1;e_1) - zq\,S(2;\delta_1) + zq^2 S(2;2\delta_1)$$
$$- zq\left(S(1;\delta_1) - S(2;\delta_1) + zq^2\,S(2;2\delta_1)\right)$$

$i = 0$ equation

$$
\begin{aligned}
&\mathrm{rel}_0(0;\vec{0}) + \mathrm{rel}_1(1;\vec{0}) - zq\,\mathrm{rel}_0(1;\delta_1) \\
&= S(0;\vec{0}) - S(1;\vec{0}) + zq\,S(1;\delta_1) \\
&+ S(1;\vec{0}) - S(1;e_1) - zq\,S(2;\delta_1) + zq^2 S(2;2\delta_1) \\
&- zq\left(S(1;\delta_1) - S(2;\delta_1) + zq^2\,S(2;2\delta_1)\right) \\
&= S(0;\vec{0}) - S(1;e_1) + zq^2 S(2;2\delta_1) - z^2q^3\,S(2;2\delta_1) \\
&= 0.
\end{aligned}
$$

Thus, we have verified that the $i = 0$ equation is satisfied.

Translating into S notation

We obtain:

$$S(0;\vec{0}) + zq^2(1-zq)S(2;2\delta_1) = S(1;e_1)$$

$$S(0;-\eta_i) + (1-zq)(1+zq^2)S(2;2\delta_1-\eta_i)$$
$$= S(1;\delta_1-\eta_{i-1}) + S(1;\delta_1-\eta_{i+1}), \quad 1 \le i < \frac{\ell-1}{2}$$

$$S(0;-\eta_{\ell/2}) + (1-zq)(1+zq^2)S(2;2\delta_1-\eta_{\ell/2})$$
$$= 2S(1;\delta_1-\eta_{\ell/2-1}), \quad \text{if } \ell \text{ is even}$$

$$S(0;-\eta_{\frac{\ell-1}{2}}) + (1-zq)(1+zq^2)S(2;2\delta_1-\eta_{\frac{\ell-1}{2}})$$
$$= S(1;\delta_1-\eta_{\frac{\ell-3}{2}}) + S(1;\delta_1-\eta_{\frac{\ell-1}{2}}), \quad \text{if } \ell \text{ is odd}$$

$1 \leq i < \ell/2$ equation

Let $1 \leq i < \ell/2$. Then

$$\begin{aligned}
&\mathrm{rel}_0(0; -\eta_i) - \mathrm{rel}_0(1; 2\delta_1 - \eta_i) - zq\,\mathrm{rel}_0(1; \delta_1 - \eta_i) \\
&+ \mathrm{rel}_1(1; -\eta_i) + \mathrm{rel}_2(1; e_1 - \eta_i) \\
&- \sum_{j=1}^{2i-1} \left(\mathrm{rel}_j\left(1; \delta_1 - \eta_i + \delta_{j+1}\right) - \mathrm{rel}_{j+2}\left(1; \delta_1 - \eta_i - \delta_{j+2}\right)\right)
\end{aligned}$$

$1 \le i < \ell/2$ equation

Let $1 \le i < \ell/2$. Then

$$\begin{aligned}
&\mathrm{rel}_0(0; -\eta_i) - \mathrm{rel}_0(1; 2\delta_1 - \eta_i) - zq\,\mathrm{rel}_0(1; \delta_1 - \eta_i) \\
&+ \mathrm{rel}_1(1; -\eta_i) + \mathrm{rel}_2(1; e_1 - \eta_i) \\
&- \sum_{j=1}^{2i-1} \left(\mathrm{rel}_j\left(1; \delta_1 - \eta_i + \delta_{j+1}\right) - \mathrm{rel}_{j+2}\left(1; \delta_1 - \eta_i - \delta_{j+2}\right)\right) \\
&= \cdots
\end{aligned}$$

$1 \le i < \ell/2$ equation

Let $1 \le i < \ell/2$. Then

$$
\begin{aligned}
&\mathrm{rel}_0(0; -\eta_i) - \mathrm{rel}_0(1; 2\delta_1 - \eta_i) - zq\,\mathrm{rel}_0(1; \delta_1 - \eta_i) \\
&+ \mathrm{rel}_1(1; -\eta_i) + \mathrm{rel}_2(1; e_1 - \eta_i) \\
&- \sum_{j=1}^{2i-1} \left(\mathrm{rel}_j\left(1; \delta_1 - \eta_i + \delta_{j+1}\right) - \mathrm{rel}_{j+2}\left(1; \delta_1 - \eta_i - \delta_{j+2}\right)\right) \\
&= \cdots \\
&= \cdots
\end{aligned}
$$

$1 \leq i < \ell/2$ equation

Let $1 \leq i < \ell/2$. Then

$$\begin{aligned}
&\mathrm{rel}_0(0; -\eta_i) - \mathrm{rel}_0(1; 2\delta_1 - \eta_i) - zq\,\mathrm{rel}_0(1; \delta_1 - \eta_i) \\
&+ \mathrm{rel}_1(1; -\eta_i) + \mathrm{rel}_2(1; e_1 - \eta_i) \\
&- \sum_{j=1}^{2i-1} \left(\mathrm{rel}_j\left(1; \delta_1 - \eta_i + \delta_{j+1}\right) - \mathrm{rel}_{j+2}\left(1; \delta_1 - \eta_i - \delta_{j+2}\right)\right) \\
&= \cdots \\
&= \cdots \\
&= S(0; -\eta_i) - S(1; \delta_1 - \eta_{i-1}) - S(1; \delta_1 - \eta_{i+1}) \\
&+ (1 - zq)(1 + zq^2) S(2; 2\delta_1 - \eta_i)
\end{aligned}$$

$1 \leq i < \ell/2$ equation

Let $1 \leq i < \ell/2$. Then

$$\begin{aligned}
&\mathrm{rel}_0(0; -\eta_i) - \mathrm{rel}_0(1; 2\delta_1 - \eta_i) - zq\,\mathrm{rel}_0(1; \delta_1 - \eta_i) \\
&+ \mathrm{rel}_1(1; -\eta_i) + \mathrm{rel}_2(1; e_1 - \eta_i) \\
&- \sum_{j=1}^{2i-1} \left(\mathrm{rel}_j\left(1; \delta_1 - \eta_i + \delta_{j+1}\right) - \mathrm{rel}_{j+2}\left(1; \delta_1 - \eta_i - \delta_{j+2}\right)\right) \\
&= \cdots \\
&= \cdots \\
&= S(0; -\eta_i) - S(1; \delta_1 - \eta_{i-1}) - S(1; \delta_1 - \eta_{i+1}) \\
&+ (1 - zq)(1 + zq^2)S(2; 2\delta_1 - \eta_i) \\
&= 0.
\end{aligned}$$

Translating into S notation

We obtain:

$$S(0;\vec{0}) + zq^2(1-zq)S(2;2\delta_1) = S(1;e_1)$$

$$S(0;-\eta_i) + (1-zq)(1+zq^2)S(2;2\delta_1-\eta_i)$$
$$= S(1;\delta_1-\eta_{i-1}) + S(1;\delta_1-\eta_{i+1}), \quad 1 \le i < \frac{\ell-1}{2}$$

$$S(0;-\eta_{\ell/2}) + (1-zq)(1+zq^2)S(2;2\delta_1-\eta_{\ell/2})$$
$$= 2S(1;\delta_1-\eta_{\ell/2-1}), \quad \text{if } \ell \text{ is even}$$

$$S(0;-\eta_{\frac{\ell-1}{2}}) + (1-zq)(1+zq^2)S(2;2\delta_1-\eta_{\frac{\ell-1}{2}})$$
$$= S(1;\delta_1-\eta_{\frac{\ell-3}{2}}) + S(1;\delta_1-\eta_{\frac{\ell-1}{2}}), \quad \text{if } \ell \text{ is odd}$$

Kanade and MCR: "Completing the A_2 Andrews-Schilling-Warnaar identities."

Proved identities like

$$\sum_{r_1,r_2,s_1,s_2\geq 0} \frac{q^{r_1^2-r_1s_1+s_1^2+r_2^2-r_2s_2+s_2^2+s_2}(1-q^{1+r_2+s_1})}{(q;q)_{r_1-r_2}(q;q)_{s_1-s_2}(q;q)_{r_2}(q;q)_{s_2}(q;q)_{r_2+s_2+1}}$$

$$= \frac{1}{(q,q)_\infty\,(q,q,q^2,q^3,q^4,q^4,q^6,q^6,q^7,q^8,q^9,q^9;\ q^{10})_\infty}.$$

related to cylindric partitions with three rows.
Proofs: **definitely** needed computers.
In some cases, hours of computer time were required to find a linear combination that works.

Outline

- Integer partitions and Rogers–Ramanujan identities — toy example
- Atomic relations and application to a (mod 10) identity
- Cylindric partitions and applications to infinite families of identities
- **Colored partition identities (time permitting)**

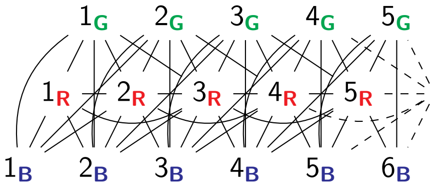
MCR. "On a pair..."

Let Γ be the set of partitions into three colors (**R**, **G**, and **B**), subject to the following conditions:

- The sizes of all parts must be distinct. (For all j, there is at most one total copy of $j_{\mathbf{R}}$, $j_{\mathbf{G}}$, and $j_{\mathbf{B}}$.)
- The following subpartitions are forbidden for all j:
  - ⋆ $j_{\mathbf{R}} + (j+1)_{\mathbf{R}}$
  - ⋆ $j_{\mathbf{R}} + (j+2)_{\mathbf{R}}$
  - ⋆ $j_{\mathbf{R}} + (j+1)_{\mathbf{B}}$
  - ⋆ $j_{\mathbf{G}} + (j+1)_{\mathbf{R}}$
  - ⋆ $j_{\mathbf{G}} + (j+2)_{\mathbf{R}}$
  - ⋆ $j_{\mathbf{G}} + (j+1)_{\mathbf{B}}$
  - ⋆ $j_{\mathbf{B}} + (j+1)_{\mathbf{R}}$
  - ⋆ $j_{\mathbf{B}} + (j+1)_{\mathbf{G}}$

Colored partitions

Lines connect forbidden pairs of parts:



Difference conditions as a matrix (read by rows):

	R	G	B
R	3	1	2
G	3	1	2
B	2	2	1

If there is a part $j_{\mathbf{R}}$, then the next largest possible red part would be $(j+3)_{\mathbf{R}}$, the next largest possible green part would be $(j+1)_{\mathbf{G}}$, and the next largest possible blue part would be $(j+2)_{\mathbf{B}}$.

Colored partitions

Goal:

Theorem

Let $A(n)$ be the number of three-colored partitions in Γ that sum to a non-negative integer n with no occurrences of $1_{\mathbf{R}}$.
Then,

$$\sum_{n\geq 0} A(n)q^n = \frac{1}{(q;q^2)_\infty (q^1,q^4;q^5)_\infty}.$$

Let $A^*(n)$ be the number of three-colored partitions in Γ that sum to a non-negative integer n with no occurrences of $1_{\mathbf{R}}$, $2_{\mathbf{R}}$, and $1_{\mathbf{B}}$.
Then,

$$\sum_{n\geq 0} A^*(n)q^n = \frac{1}{(q;q^2)_\infty (q^2,q^3;q^5)_\infty}.$$

Multisums

Define, for nonnegative integers a, b, c,

$$S_{a,b,c}(x) = \sum_{i,j,k \geq 0} \frac{q^{P(a,b,c,i,j,k)}}{(q;q)_{i+j+k}} \begin{bmatrix} i+j+k \\ i,\ j,\ k \end{bmatrix}_{q^2} x^{2i+j+k}$$

$$T_{a,b,c}(x) = \sum_{i,j,k \geq 0} \frac{q^{P(a,b,c,i,j,k)}}{(q;q)_{i+j+k}} \begin{bmatrix} i+j+k \\ i,\ j,\ k \end{bmatrix}_{q^2} x^{i+j+k}$$

$$P(a,b,c,i,j,k) = \frac{3}{2}i^2 + \frac{1}{2}j^2 + \frac{1}{2}k^2 + ij + ik + jk - \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k + ai + bj + ck$$

q-trinomial coefficient: $\begin{bmatrix} i+j+k \\ i,\ j,\ k \end{bmatrix}_q = \frac{(q;q)_{i+j+k}}{(q;q)_i\,(q;q)_j\,(q;q)_k}.$

Note that

$$S_{a,b,c}(1) = T_{a,b,c}(1).$$

Atomic relations

Sample atomic relations for $S_{a,b,c}$ are:

$$\begin{aligned}
\mathrm{rel}^1_{a,b,c}: \quad & S_{a,b,c} - S_{a+2,b,c} - x^2 q^{a+1} S_{a+3,b+1,c+1} \\
& - x^2 q^{a+2} S_{a+4,b+2,c+2} && = 0 \\
\mathrm{rel}^4_{a,b,c}: \quad & S_{a,b,c} - S_{a+1,b+1,c+1} - x^2 q^{a+1} S_{a+3,b+3,c+3} \\
& - x q^{b+1} S_{a+1,b+1,c+3} - x q^{c+1} S_{a+1,b+1,c+1} && = 0
\end{aligned}$$

Similar ones exist for $T_{a,b,c}$. Some of their proofs involve the following q-Pascal-type relation:

$$\begin{bmatrix} i+j+k \\ i,\, j,\, k \end{bmatrix}_q = q^{j+k} \begin{bmatrix} i+j+k-1 \\ i-1,\, j,\, k \end{bmatrix}_q + q^{k} \begin{bmatrix} i+j+k-1 \\ i,\, j-1,\, k \end{bmatrix}_q + \begin{bmatrix} i+j+k-1 \\ i,\, j,\, k-1 \end{bmatrix}_q$$

Bivariate generating functions

For non-negative integers m and n:

- Let $A(m, n)$ be the number of three-colored partitions in Γ of n with exactly m parts that have no occurrences of $1_{\mathbf{R}}$.
- Let $A^{\star}(m, n)$ be the number of three-colored partitions in Γ of n with exactly m parts that have no occurrences of $1_{\mathbf{R}}$, $2_{\mathbf{R}}$, and $1_{\mathbf{B}}$.

$$Q(x) = \sum_{m,n\geq 0} A(m, n)x^m q^n$$

$$Q^{\star}(x) = \sum_{m,n\geq 0} A^{\star}(m, n)x^m q^n.$$

Functional equations

Uses combinatorial arguments and the Murray-Miller algorithm:

$$Q(x) - \left(1 + xq + xq^2\right) Q(xq) - xq(1 - q - xq^3)Q(xq^2) - xq^2 \left(1 - xq^2\right) Q(xq^3) = 0$$

$$Q^\star(x) - \left(1 + xq + xq^2\right) Q^\star(xq) + x^2q^4\, Q^\star\left(xq^2\right) - xq^3 \left(1 - xq^2\right) Q^\star\left(xq^3\right) = 0$$

Proof of second identity

First, we will prove:

$$S_{2,0,1}(x) = (-xq;q)_\infty \sum_{n\geq 0} \frac{x^n q^{n^2+n}}{(q;q)_n}$$

by showing that both sides satisfy the following functional equation:

$$g(x) = (1+xq)g(xq) + xq^2(1+xq)\left(1+xq^2\right)g\left(xq^2\right)$$

Substituting $S_{2,0,1}(x)$ produces

$$S_{2,0,1} - (1+xq)S_{4,1,2} - xq^2(1+xq)\left(1+xq^2\right)S_{6,2,3} = 0.$$

This can be written as a linear combination of atomic relations.

Proof of second identity

Next, show

$$T_{2,0,1}(x) = Q^{\star}(x).$$

Substituting $T_{2,0,1}(x)$ into the functional equation for $Q^{\star}(x)$ gives

$$T_{2,0,1} - (1 + xq + xq^2)\, T_{3,1,2} + x^2 q^4\, T_{4,2,3} - xq^3(1 - xq^2)\, T_{5,3,4} = 0.$$

This can be written as a linear combination of atomic relations.

$$\begin{aligned} Q^{\star}(1) &= T_{2,0,1}(1) \\ &= S_{2,0,1}(1) \\ &= (-q;q)_\infty \sum_{n \geq 0} \frac{q^{n^2+n}}{(q;q)_n} \\ &= \frac{1}{(q;q^2)_\infty (q^2,q^3;q^5)_\infty}. \end{aligned}$$