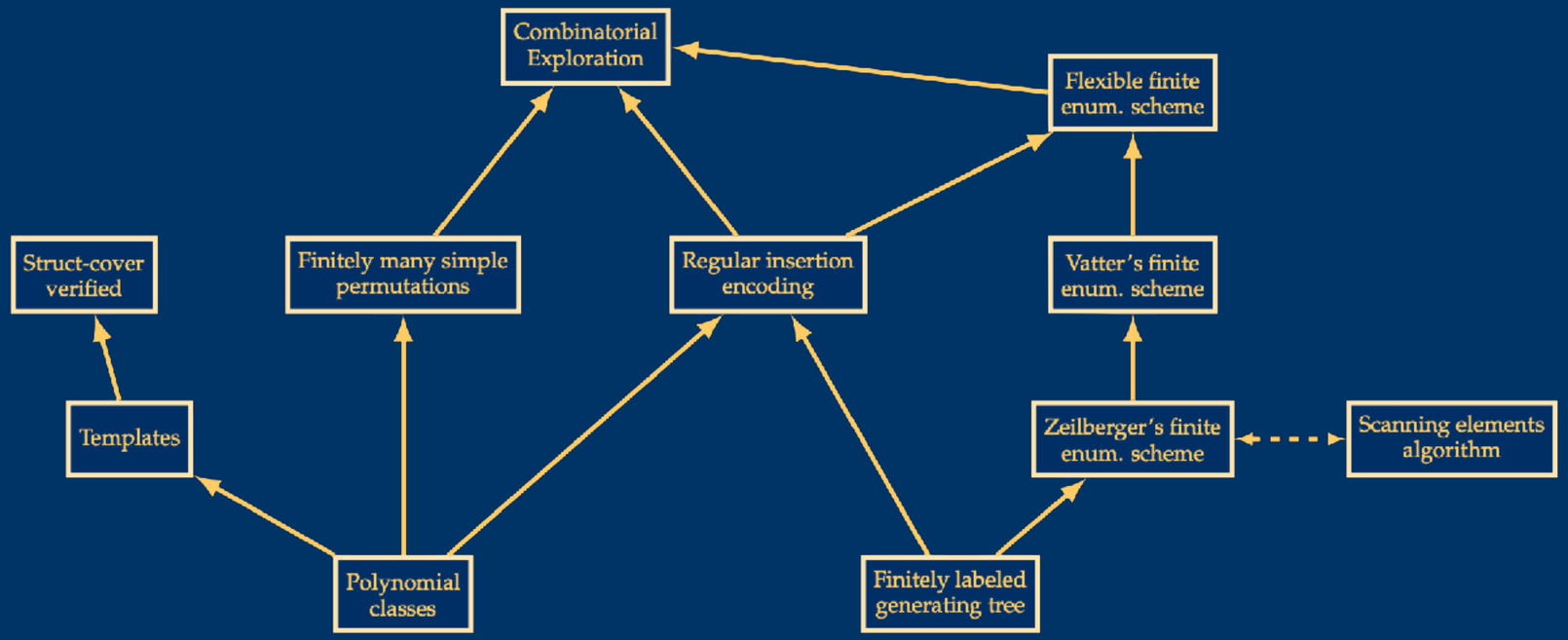
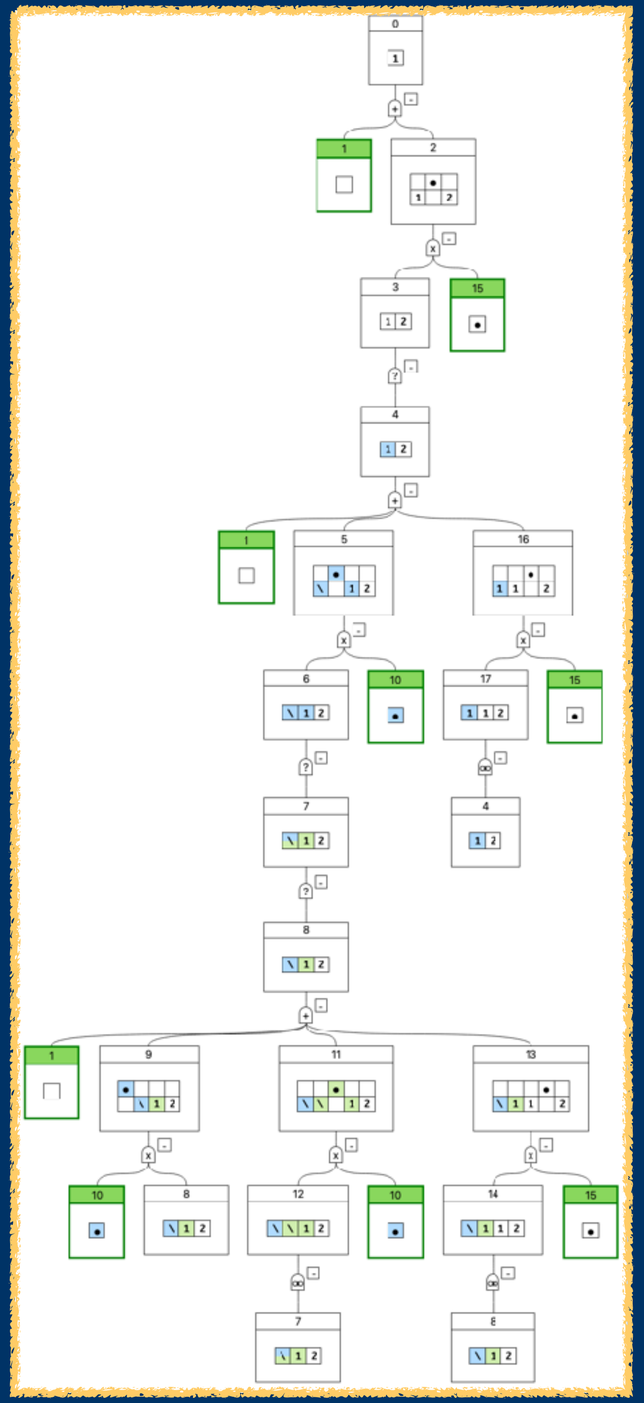
Computational and Experimental Methods in Permutation Patterns

Jay Pantone
Marquette University

Rutgers Experimental Mathematics Seminar





28 years ago...

Enumeration Schemes and, More Importantly, Their Automatic Generation

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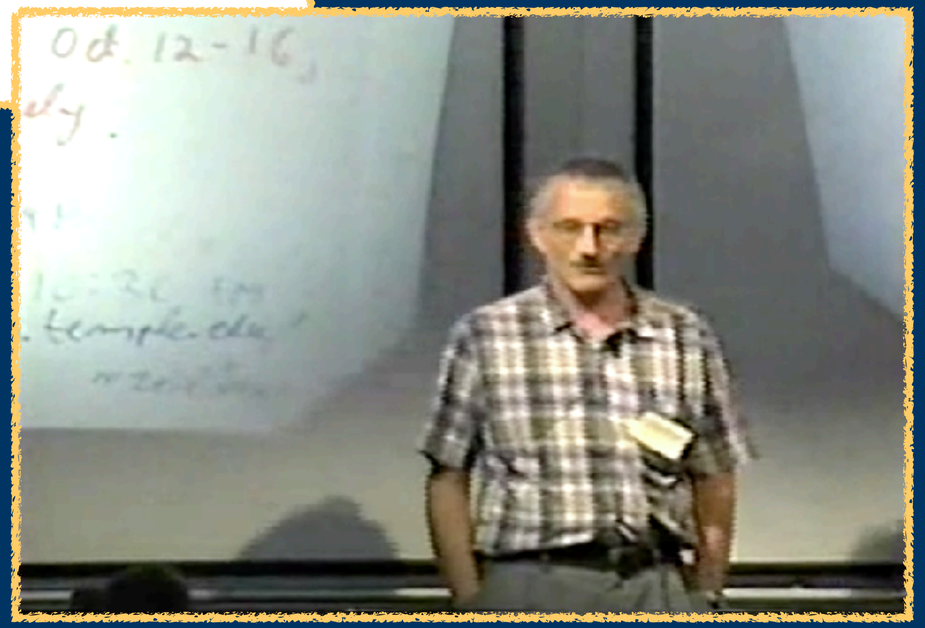

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It is way too soon to teach our computers how to become full-fledged *humans*. It is even premature to teach them how to become *mathematicians*; it is even unwise, at present, to teach them how to become *combinatorialists*. But the time is ripe to teach them how to become experts in a suitably defined and narrowly focused subarea of combinatorics. In this article, the author will describe his efforts in teaching his beloved computer, Shalosh B. Ekhad, how to be an enumerator of Wilf classes.

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With all due respect to Wilf classes and enumeration, and even to combinatorics, the main point of this article is not to enhance our understanding of Wilf classes, but to *illustrate* how much (if not all) of mathematical research will be conducted in a few years. It goes as follows. Suppose a (as of now, human) mathematician has a brilliant idea. Teach that idea to a computer and let the computer 'do research' using that idea.

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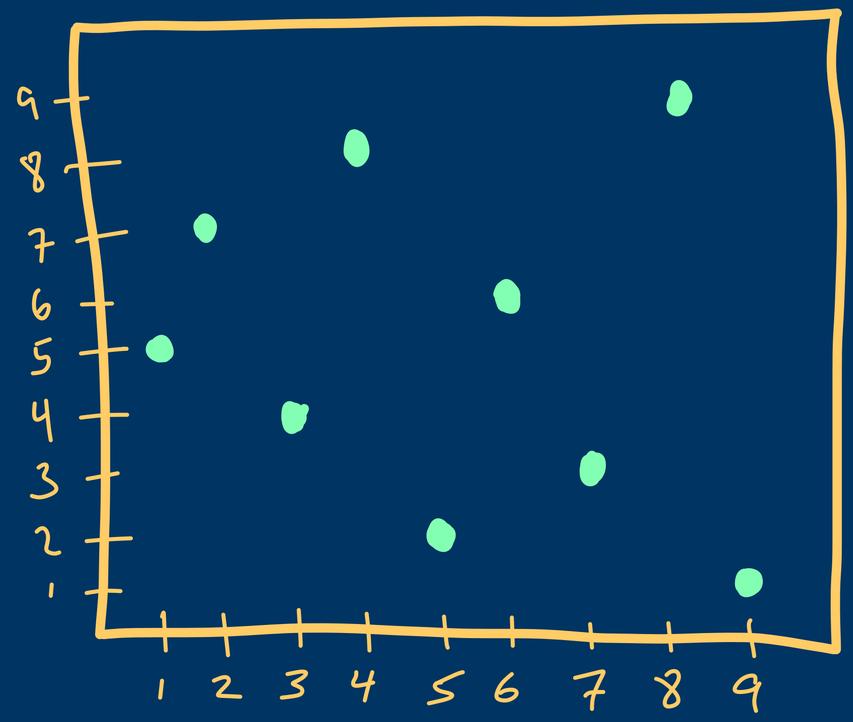
What's the state of computation and experimentation in Permutation Patterns?

Permutation Patterns

We write permutations in one-line notation:

574826391

And we plot them by plotting the points $(i, \pi(i))$:



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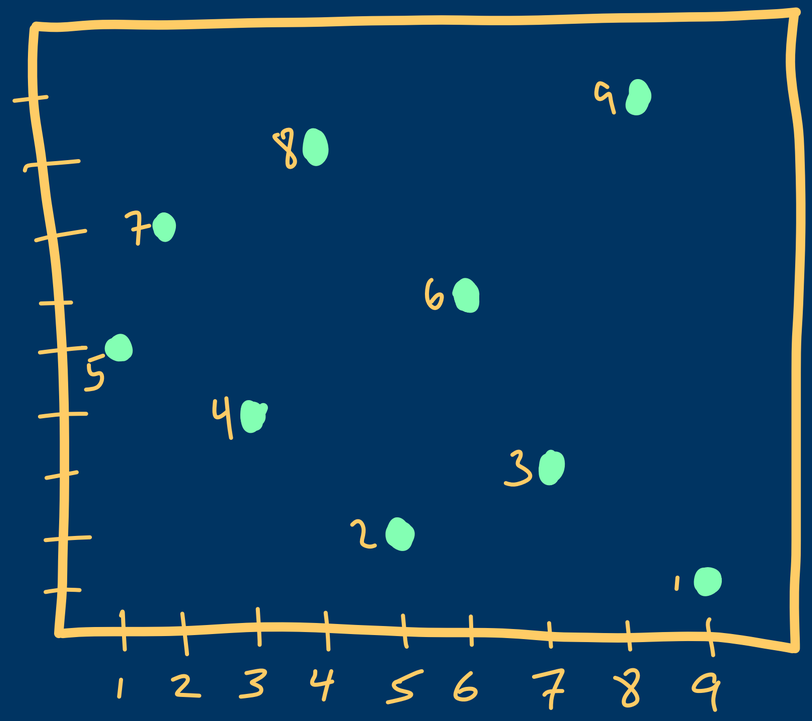

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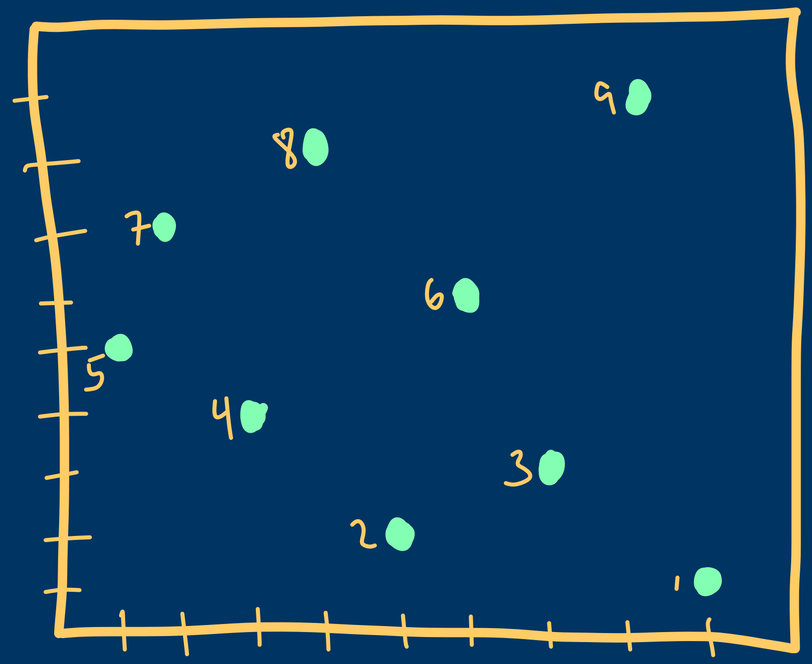

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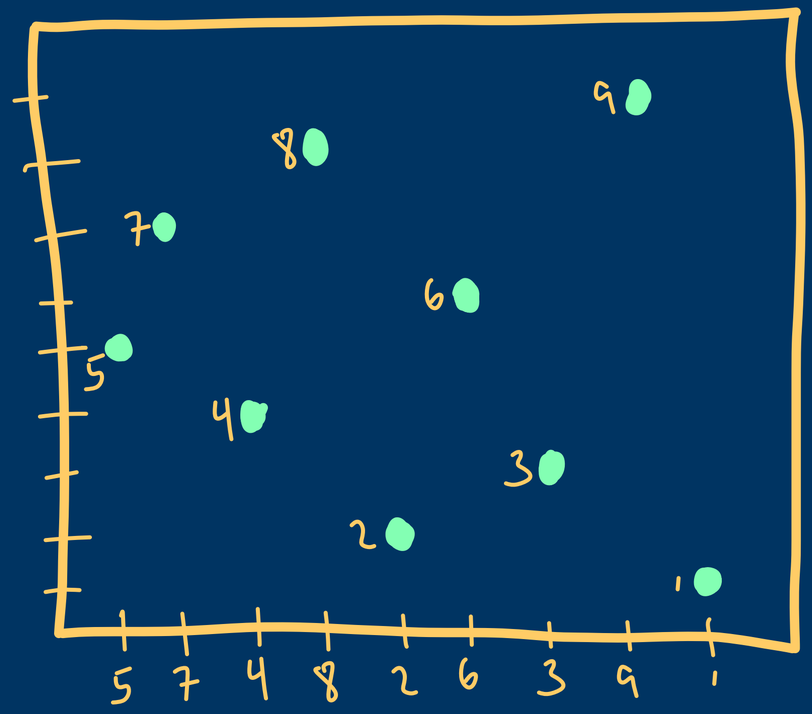
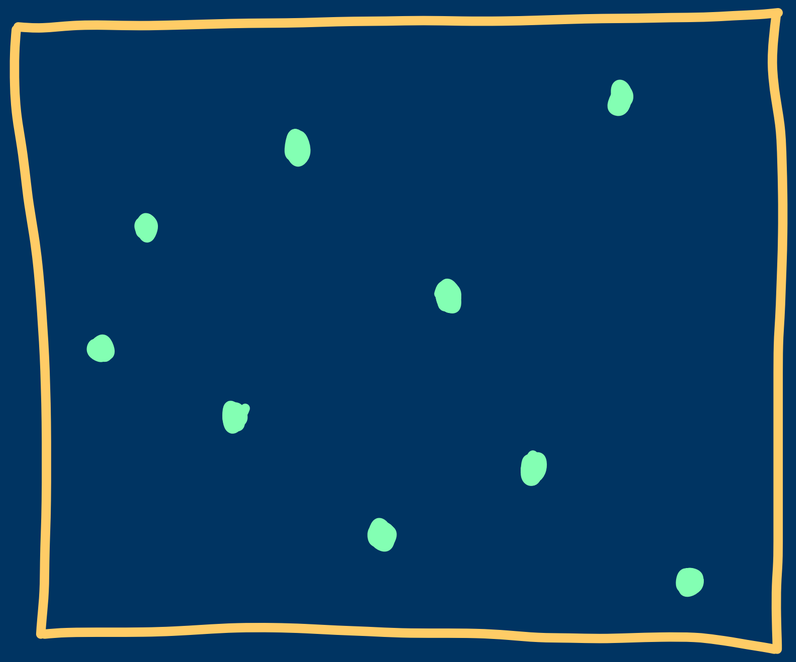
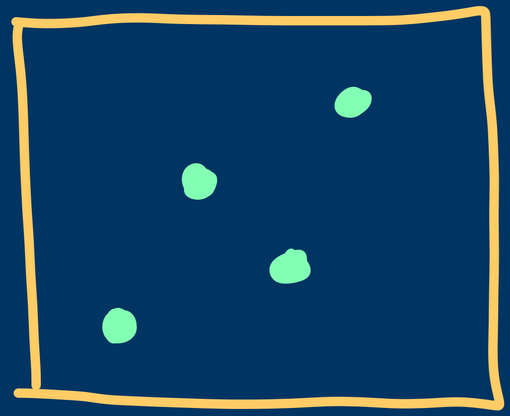


Permutation Patterns

Permutation π contains permutation σ if π has a subsequence of entries that is order-isomorphic to σ .





574826391 contains 1324

Otherwise we say π avoids σ.

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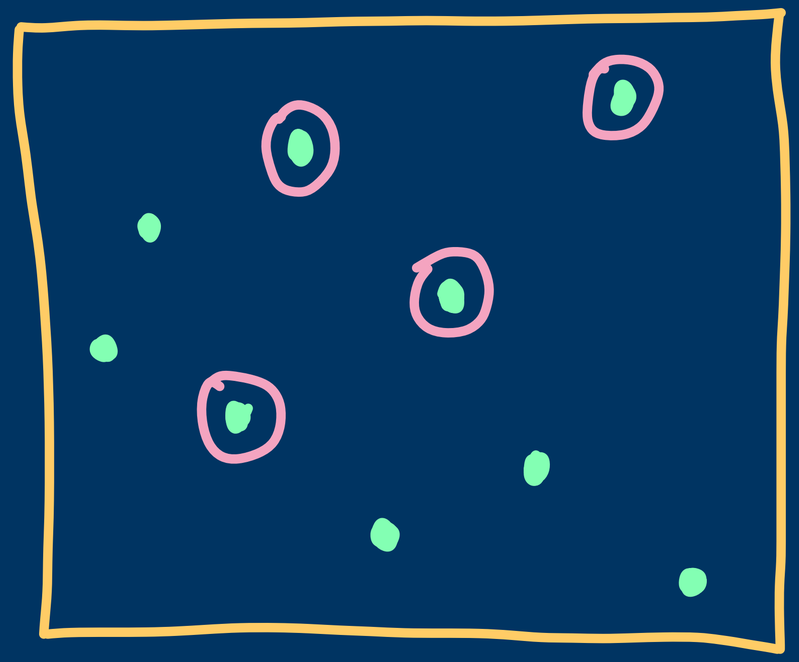

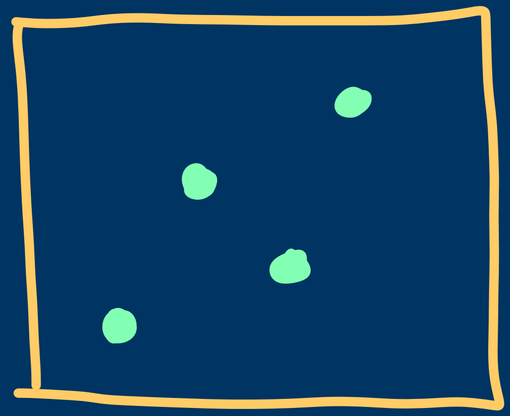

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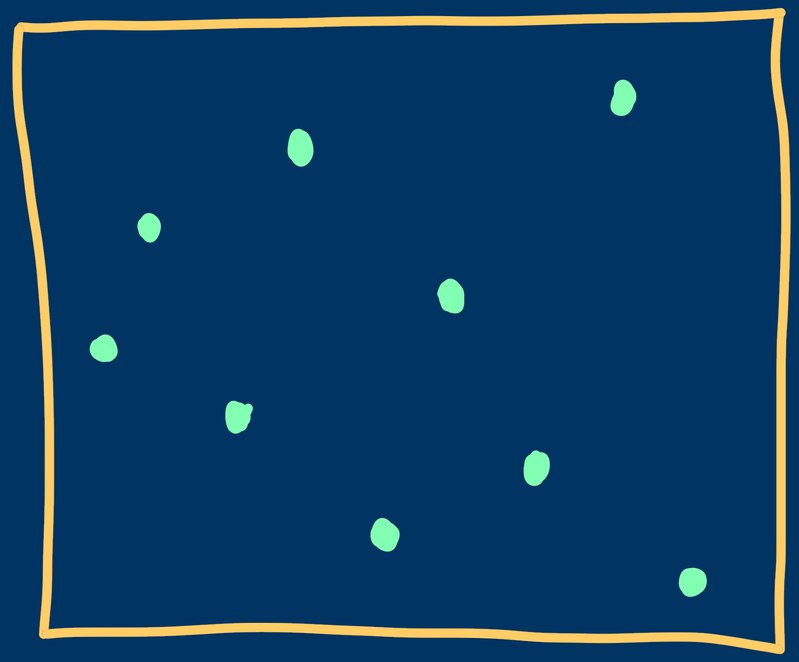

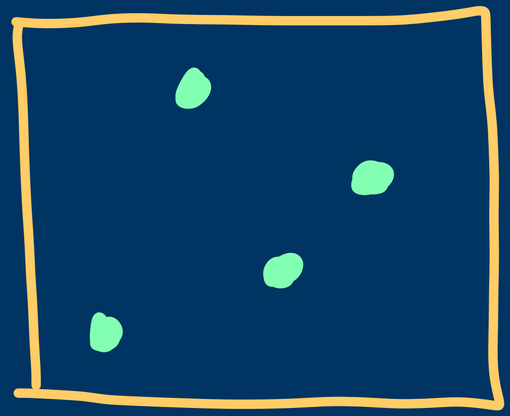

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A permutation class is a set of permutations defined by avoiding a set of patterns (permutations).

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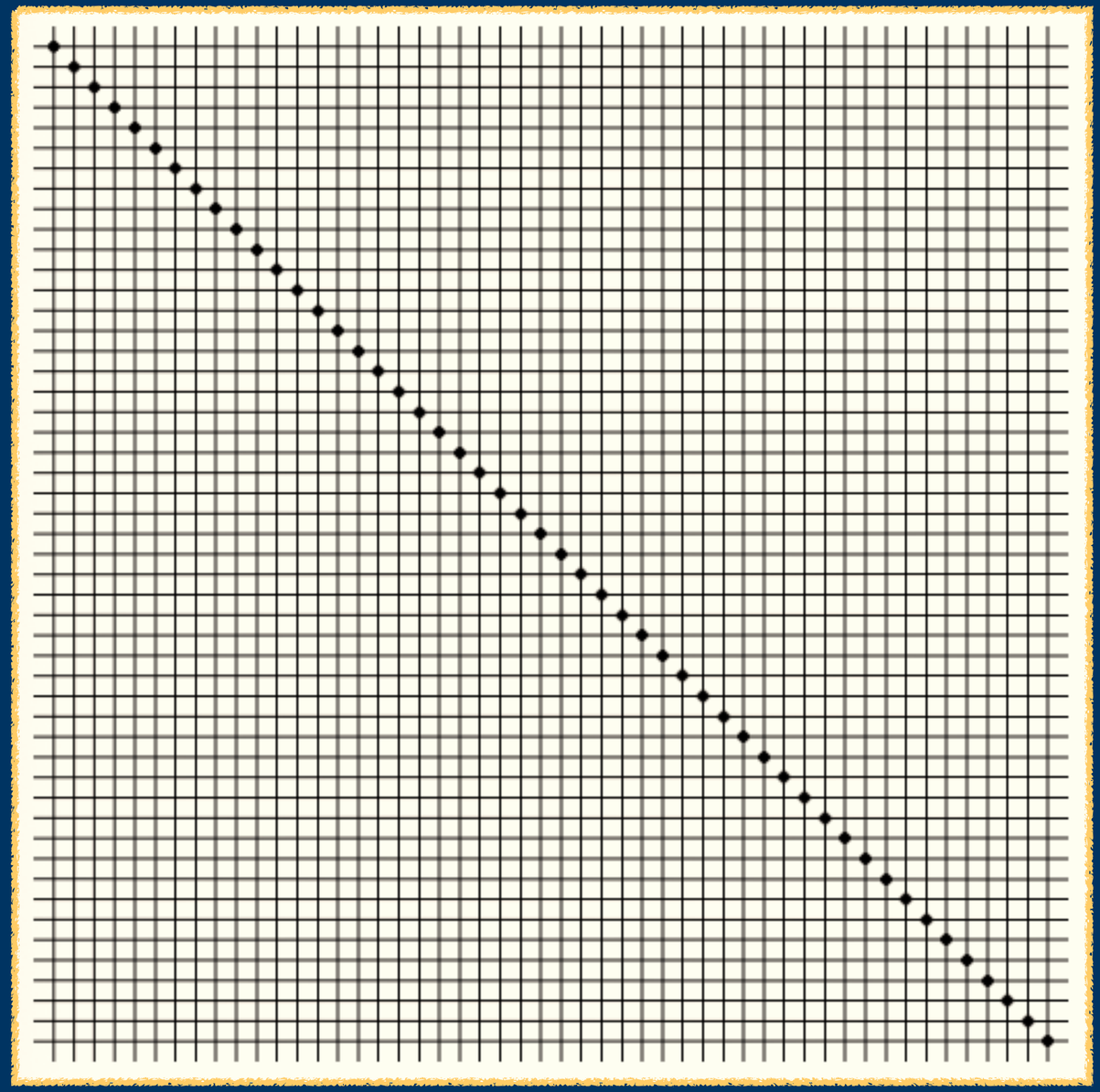
The permutation containment relation forms a poset on the set of all permutations of all lengths, and permutation classes are the downward closed sets in this poset.

Permutation Patterns

$$\mathrm{Av}(12)$$

$$a_n = 1$$

$$\frac{1}{1-x}$$

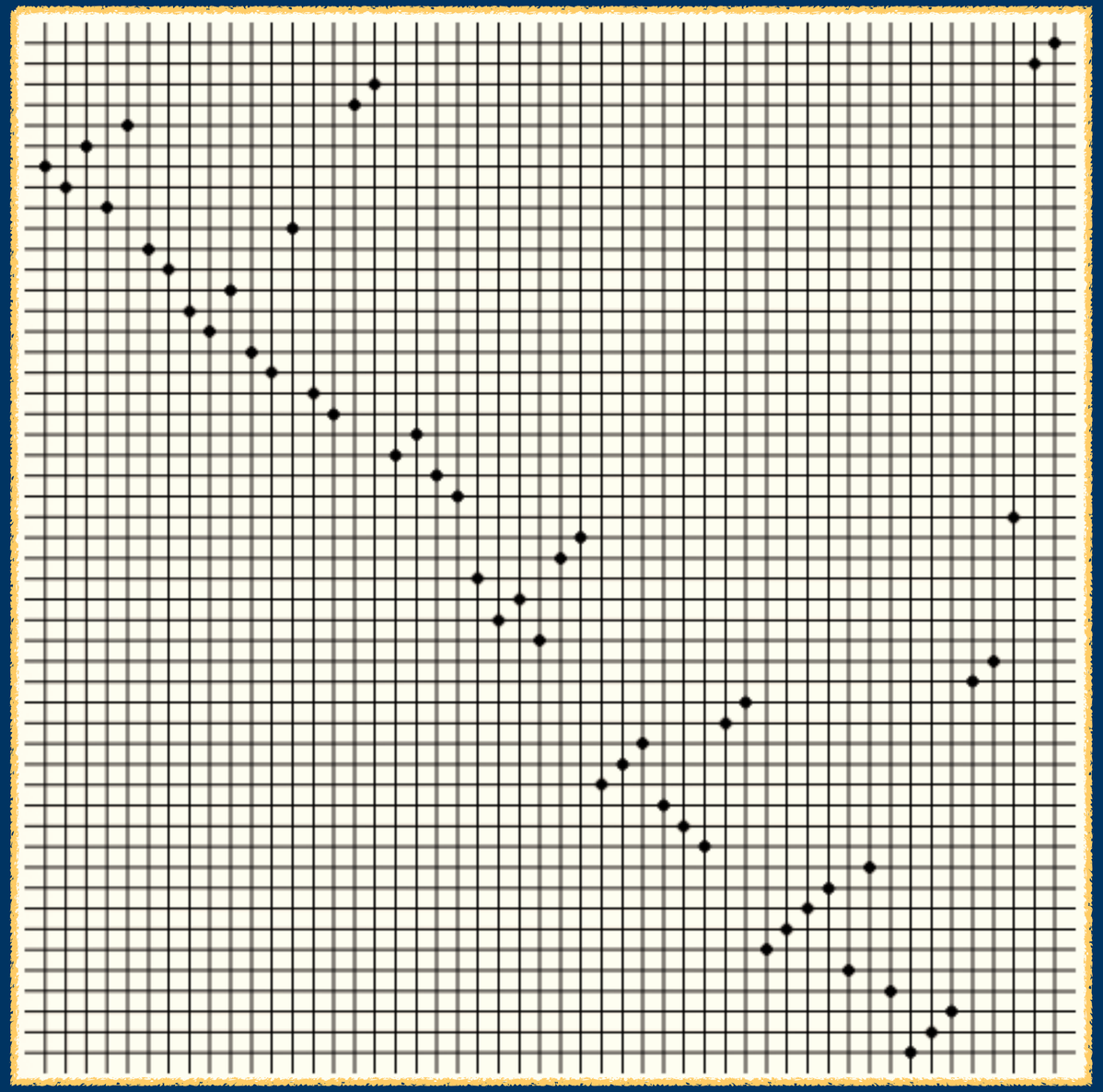


Permutation Patterns

$$\mathrm{Av}(132)$$

$$a_n = \frac{1}{n+1}\binom{2n}{n}$$

$$\frac{1-\sqrt{1-4x}}{2x}$$



Permutation Patterns

$$\mathrm{Av}(123)$$

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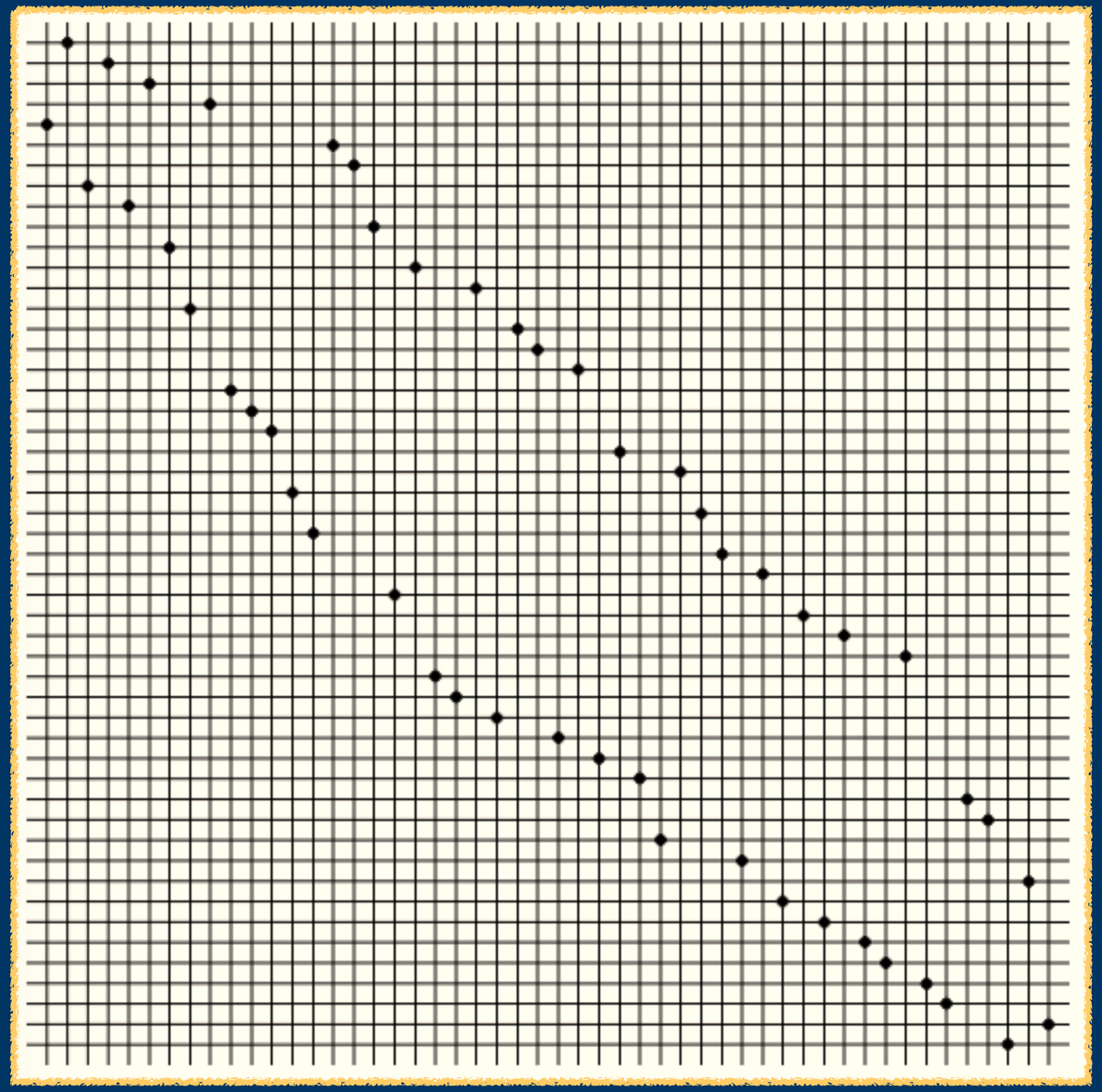
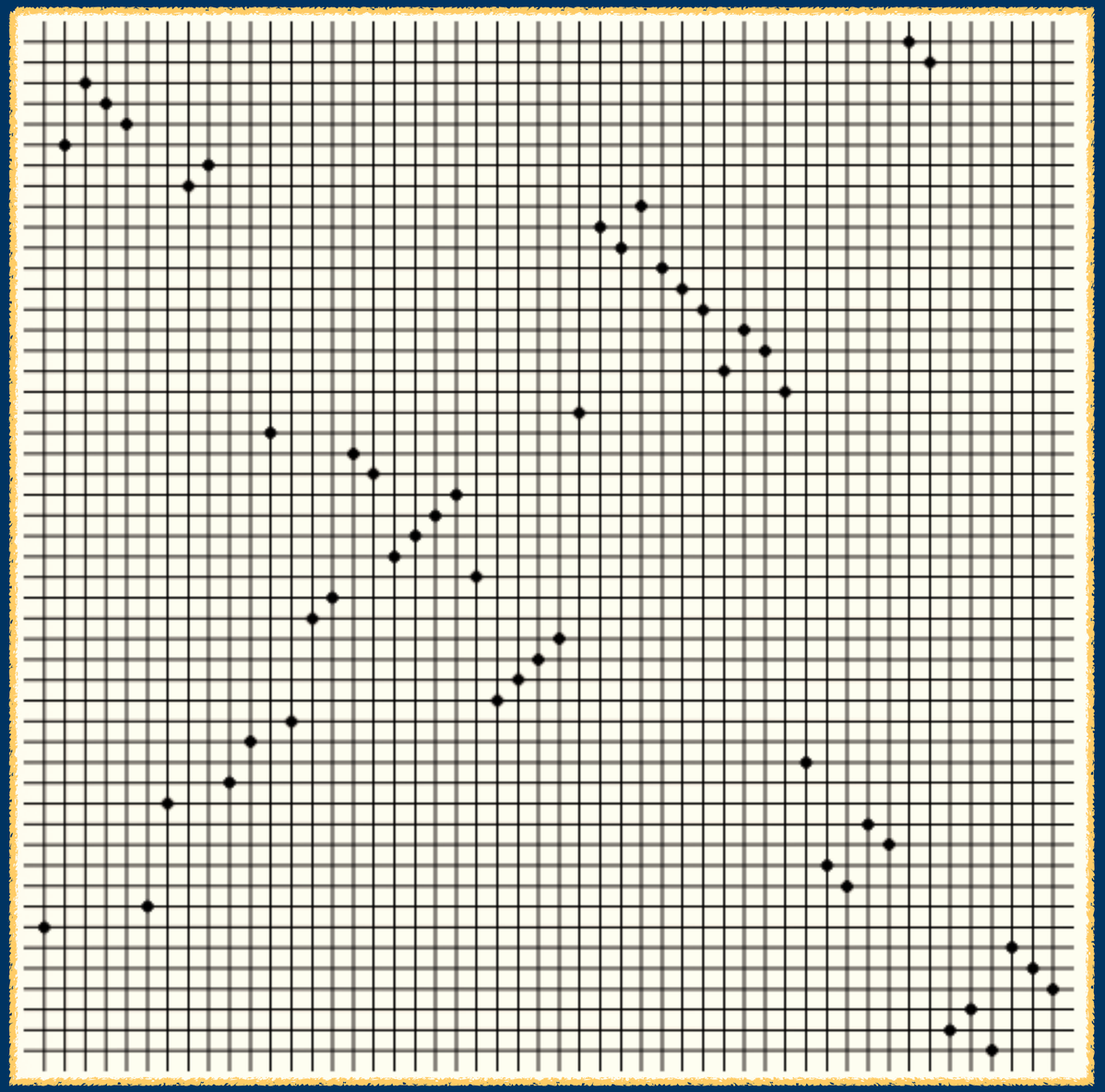


Permutation Patterns

$$\mathrm{Av}(2413, 3142)$$

$$a_n = \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{n-1}{k} \binom{n+k}{k}$$

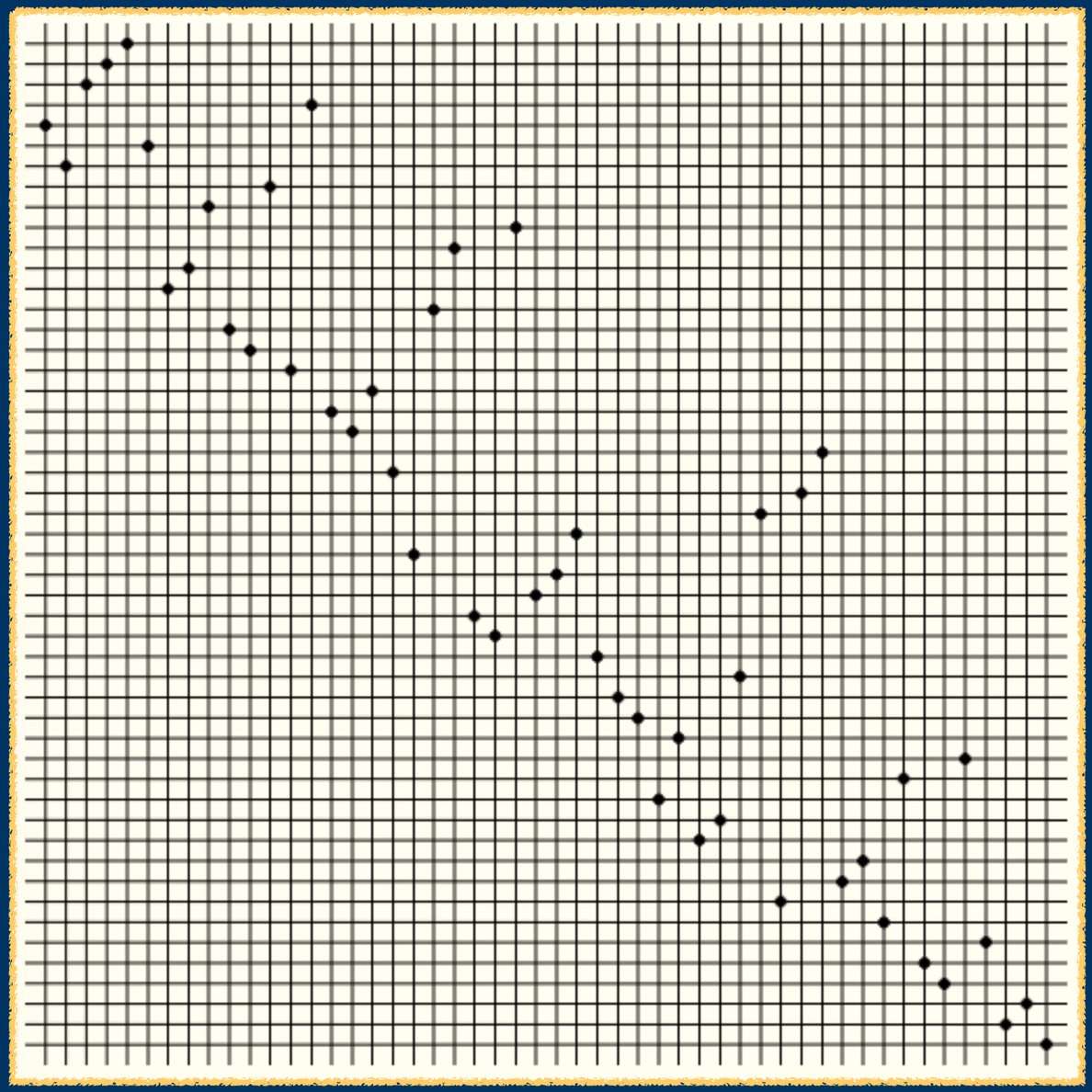
$$\frac{3 - x - \sqrt{1 - 6x + x^2}}{2}$$



Permutation Patterns

Av(1324, 1432)

???



Enumeration Schemes

Goal: Teach the computer how to search for structure in a permutation class.

Enumeration Schemes and, More Importantly, Their Automatic Generation

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Goal: Teach the computer how to search for structure in a permutation class.

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Enumeration Scheme ⟶ A polynomial-time algorithm to compute the number of permutations of length n.

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"The Most Trivial Non-Trivial Example" – $\mathrm{Av}(123)$

Define $A(n) := \mathrm{Av}_n(123)$.

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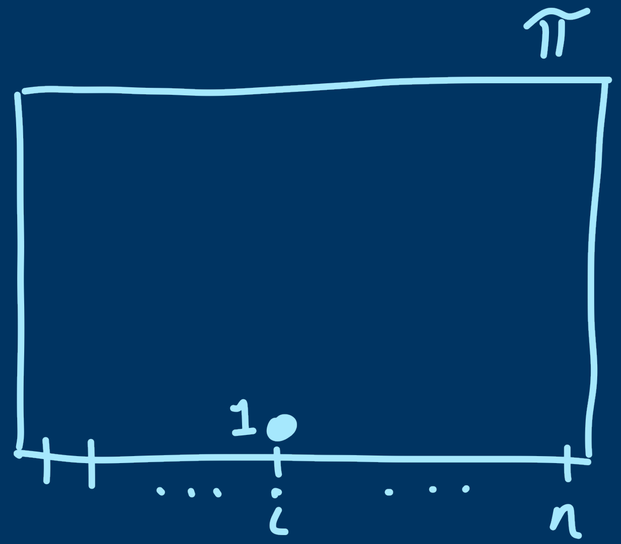

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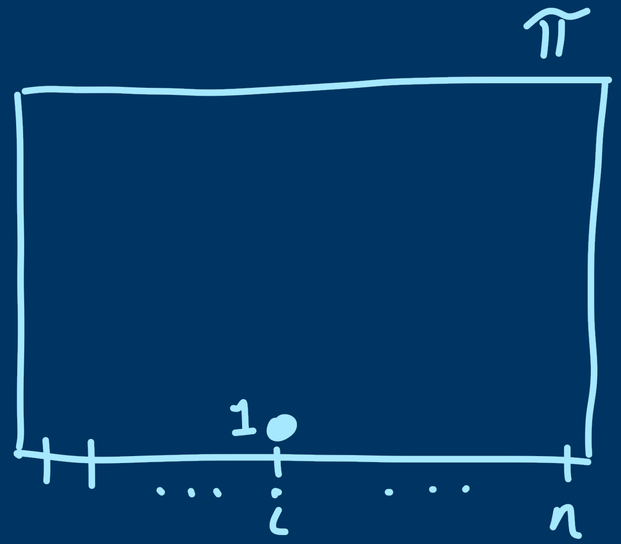
Obviously $A(n) = \bigcup_{i=1}^{n} A_1(n,i)$.



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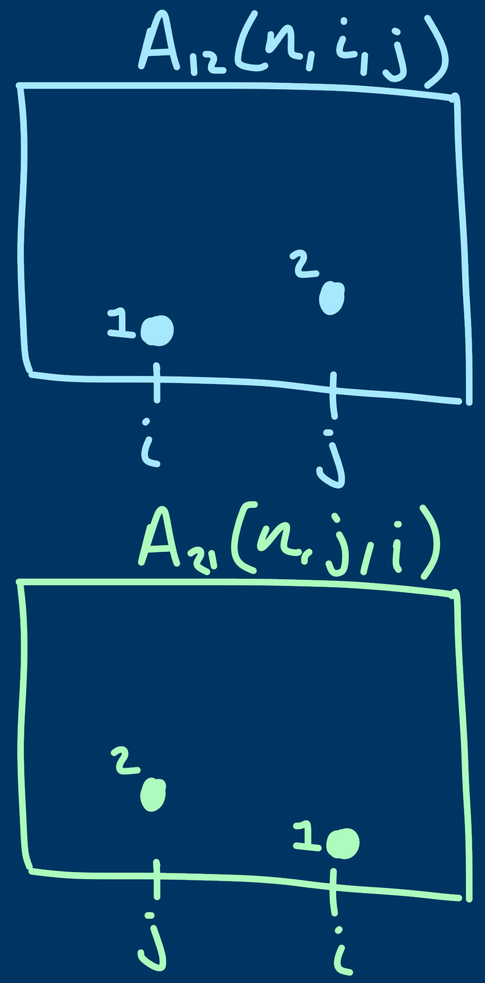
Define $A_{12}(n,i,j) := \{\pi \in A(n) : \pi(i) = 1,\ \pi(j) = 2,\ i < j\}$
and $A_{21}(n,j,i) := \{\pi \in A(n) : \pi(i) = 1,\ \pi(j) = 2,\ i > j\}$



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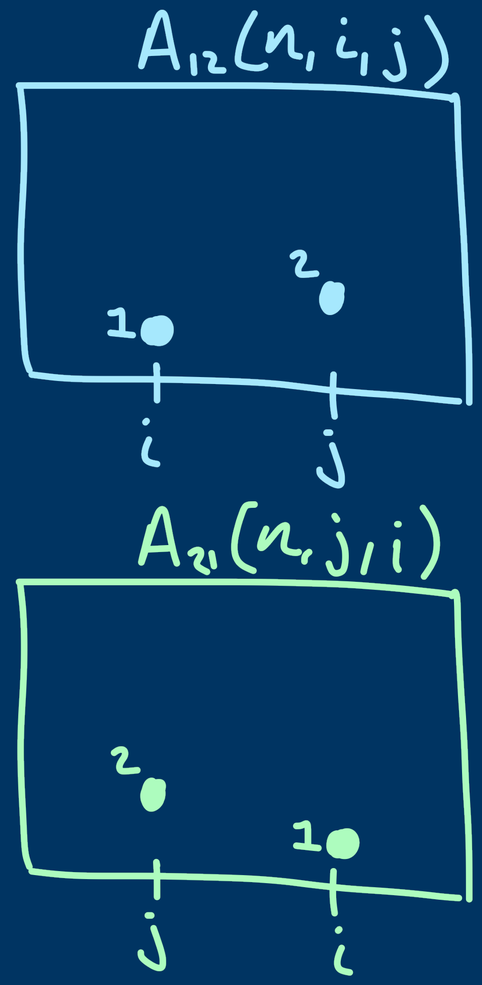
Obviously $A_1(n,i) = \left(\bigcup_{j=1}^{i-1} A_{21}(n,j,i) \right) \cup \left(\bigcup_{j=i+1}^{n} A_{12}(n,i,j) \right).$



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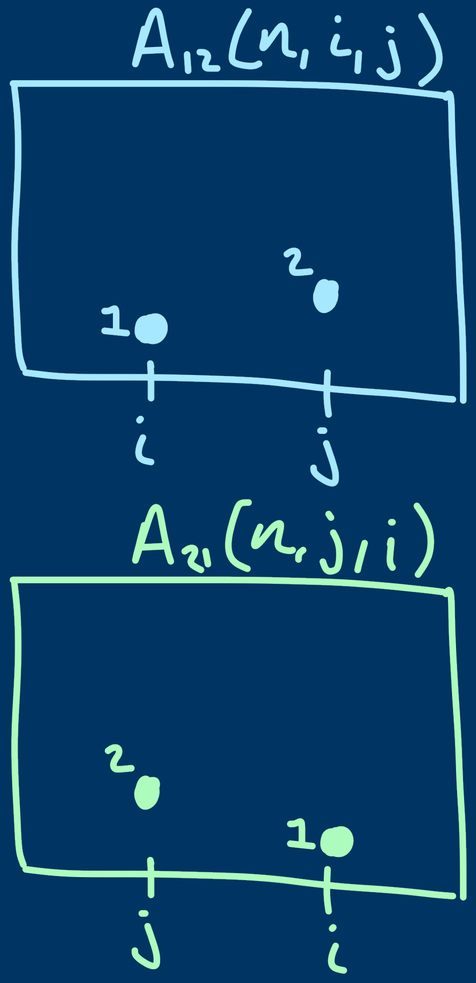
We could do this forever...



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Claim 1: $|A_{21}(n,j,i)| = |A_1(n-1,j)|$

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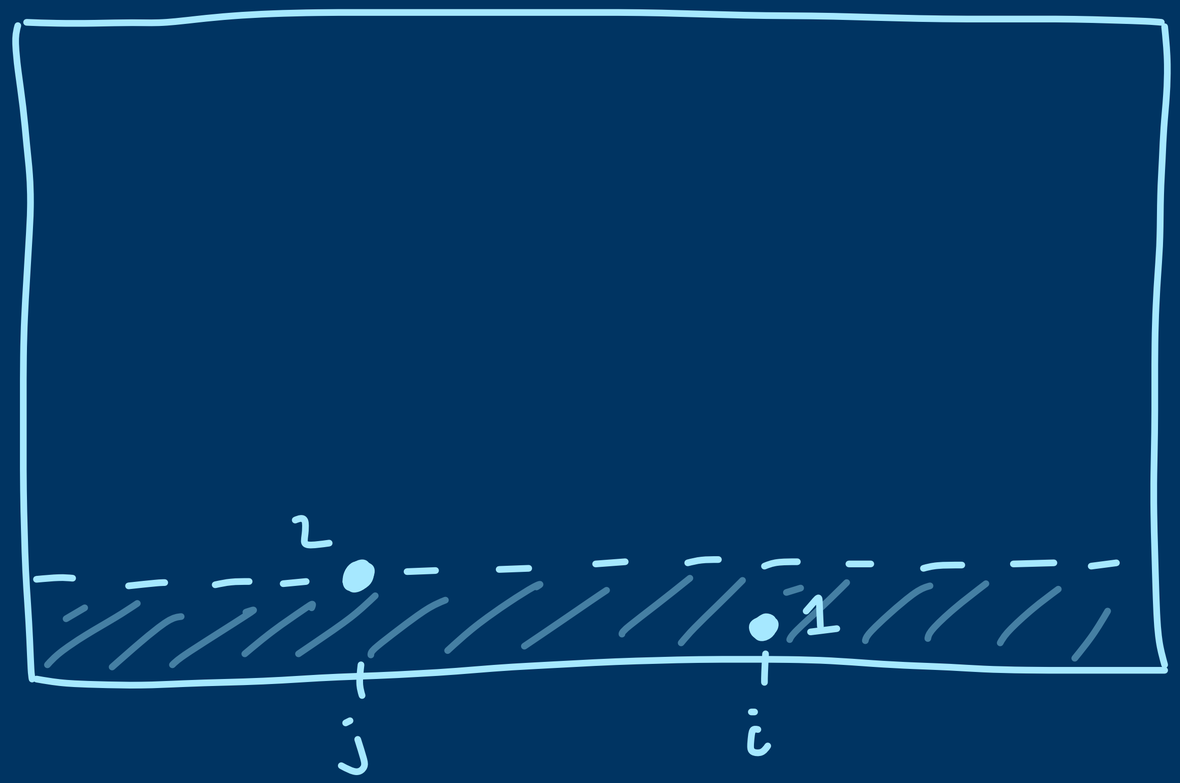

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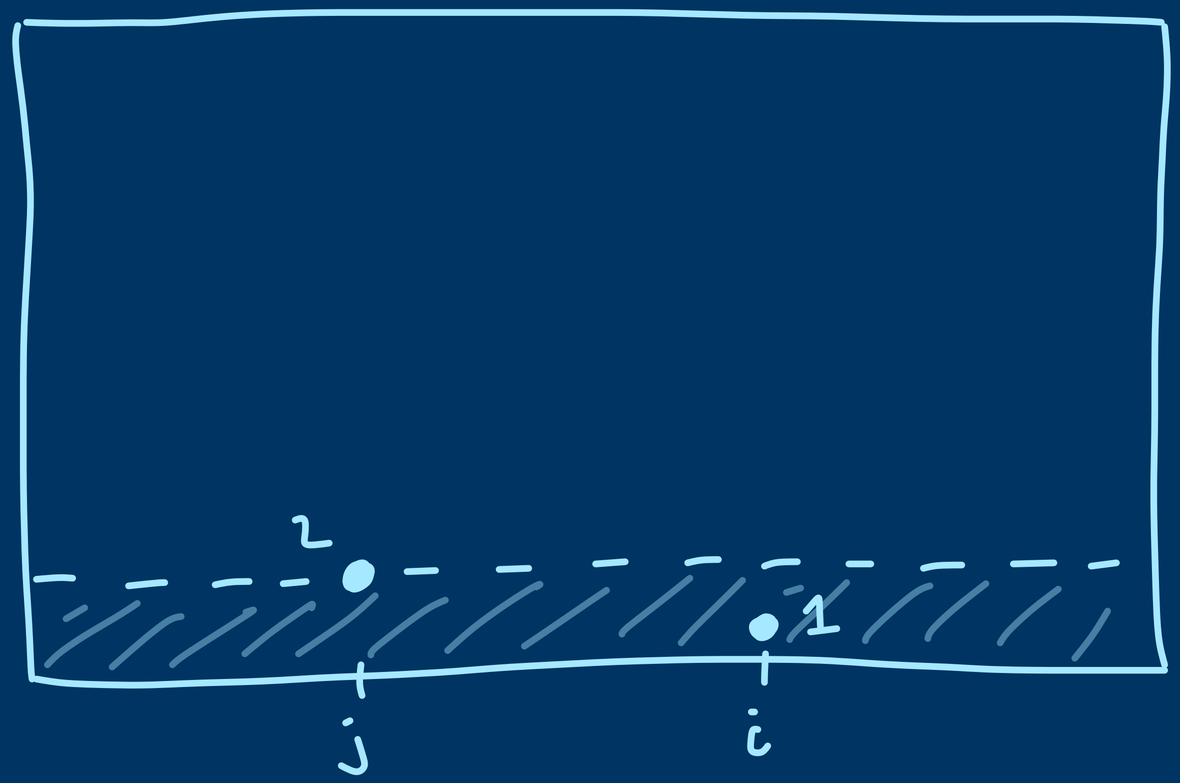

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π contains 123
iff
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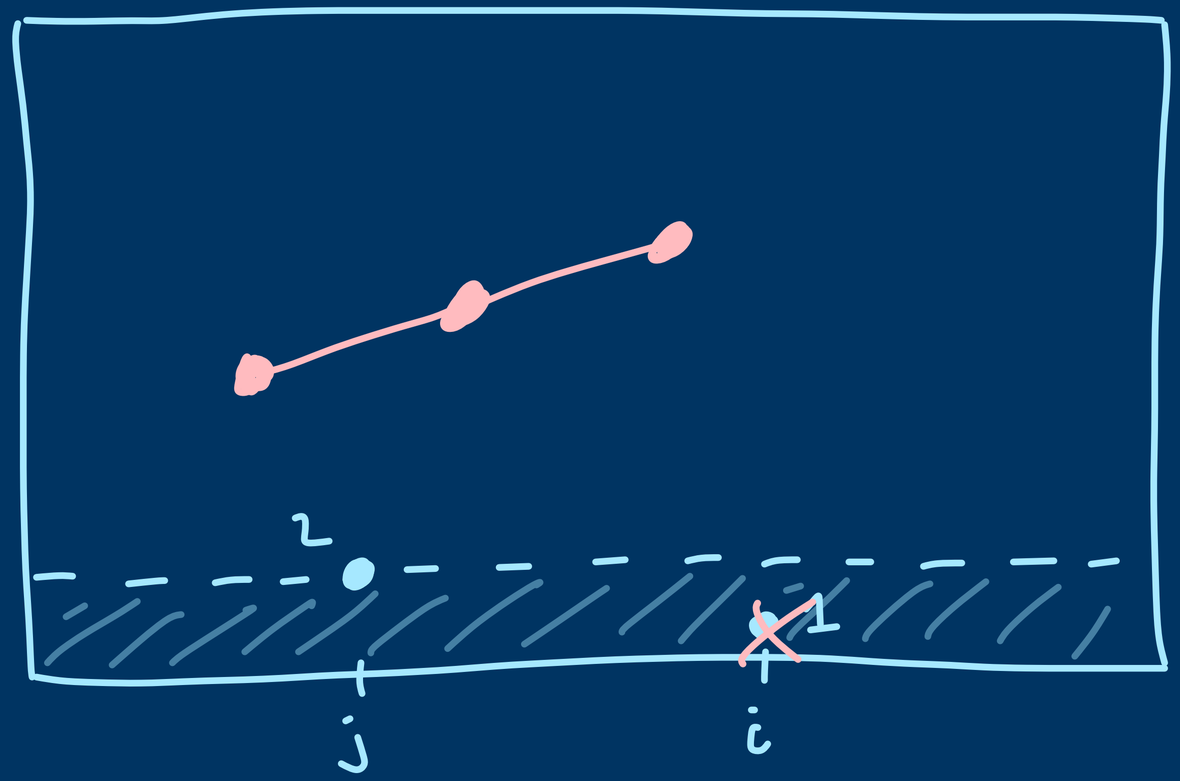

π contains 123
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Any 123 that doesn't involve $\pi(i)$ is obviously still in $\pi - \pi(i)$.

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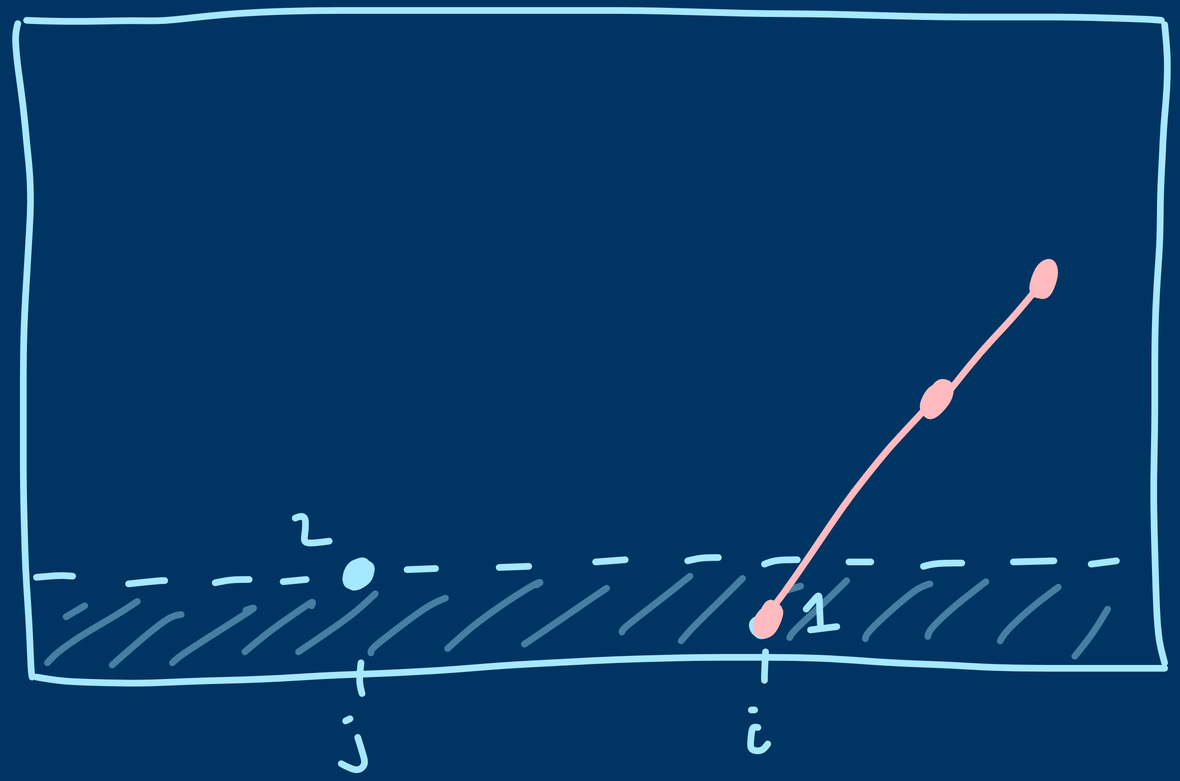

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If an occurrence of 123 involves $\pi(i)$, it must be the 1.

$\pi - \pi(i)$ has a 123 with $\pi(j)$ substituting for $\pi(i)$.

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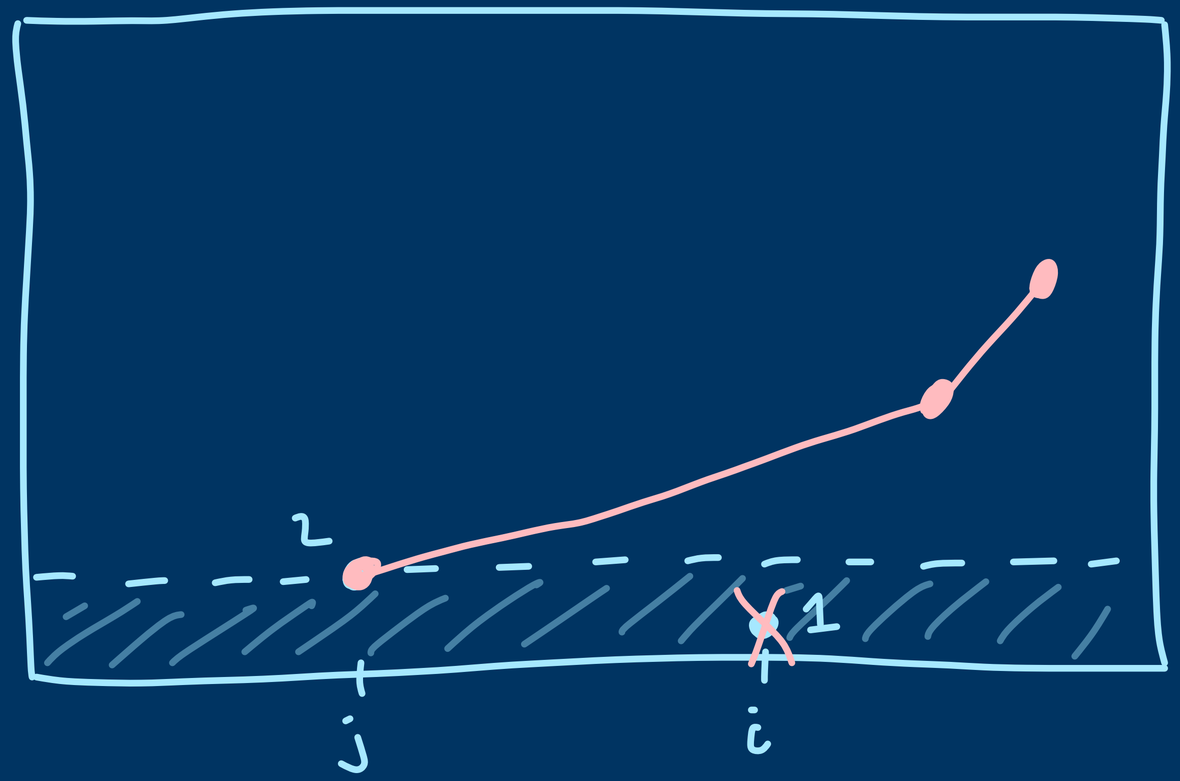

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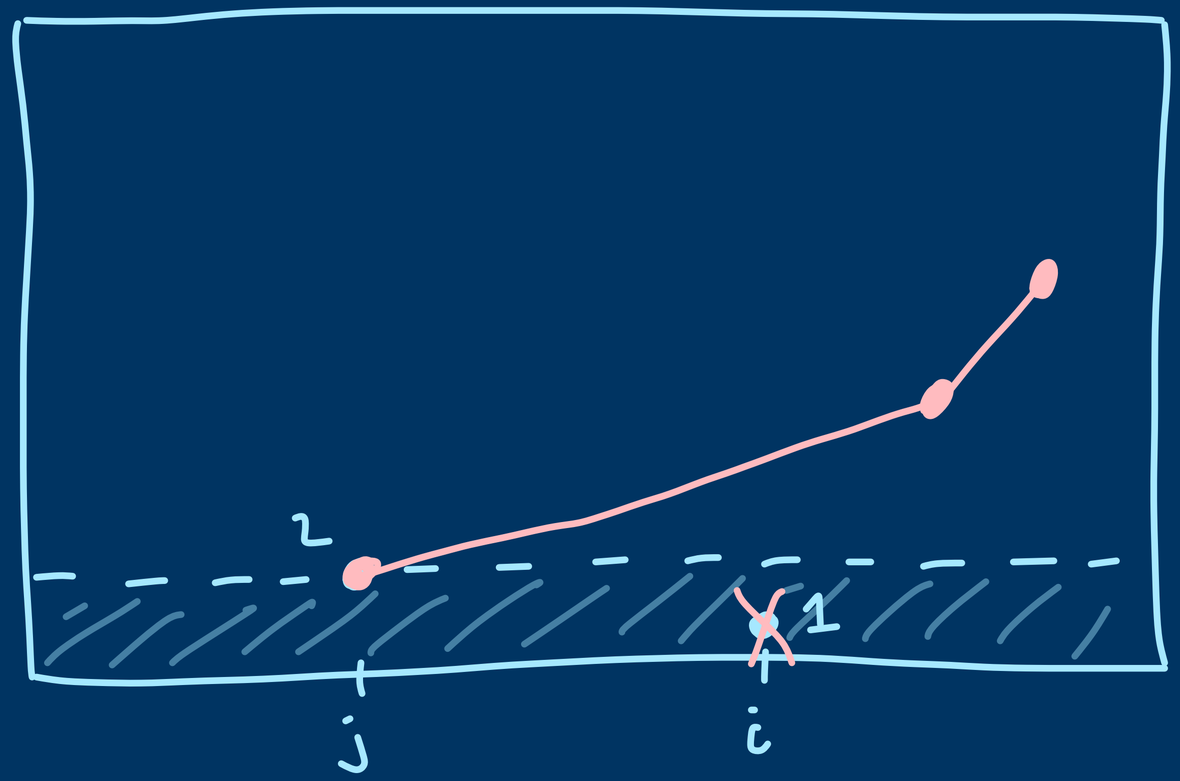
Enumeration Schemes

"The Most Trivial Non-Trivial Example" – $\mathrm{Av}(123)$

Claim 1: $|A_{21}(n,j,i)| = |A_1(n-1,j)|$



π contains 123
iff
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So there is a bijection between $A_{21}(n,j,i)$ and $A_1(n-1,j)$.

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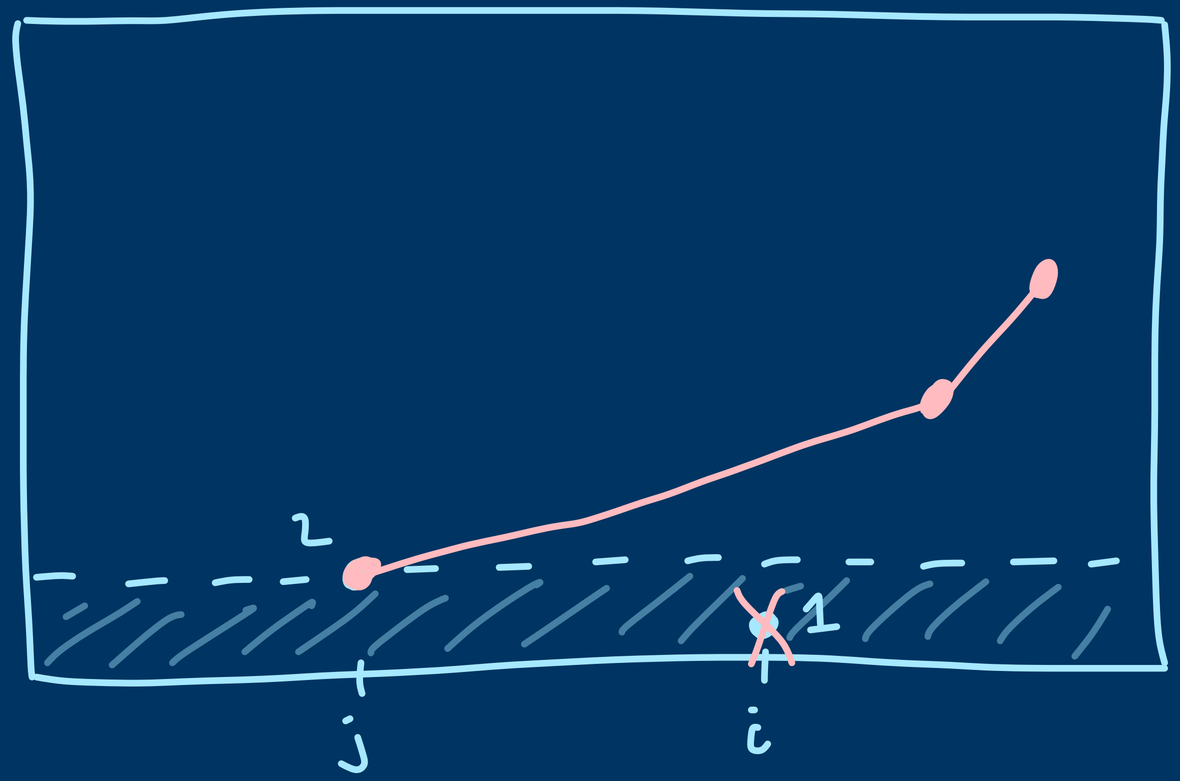

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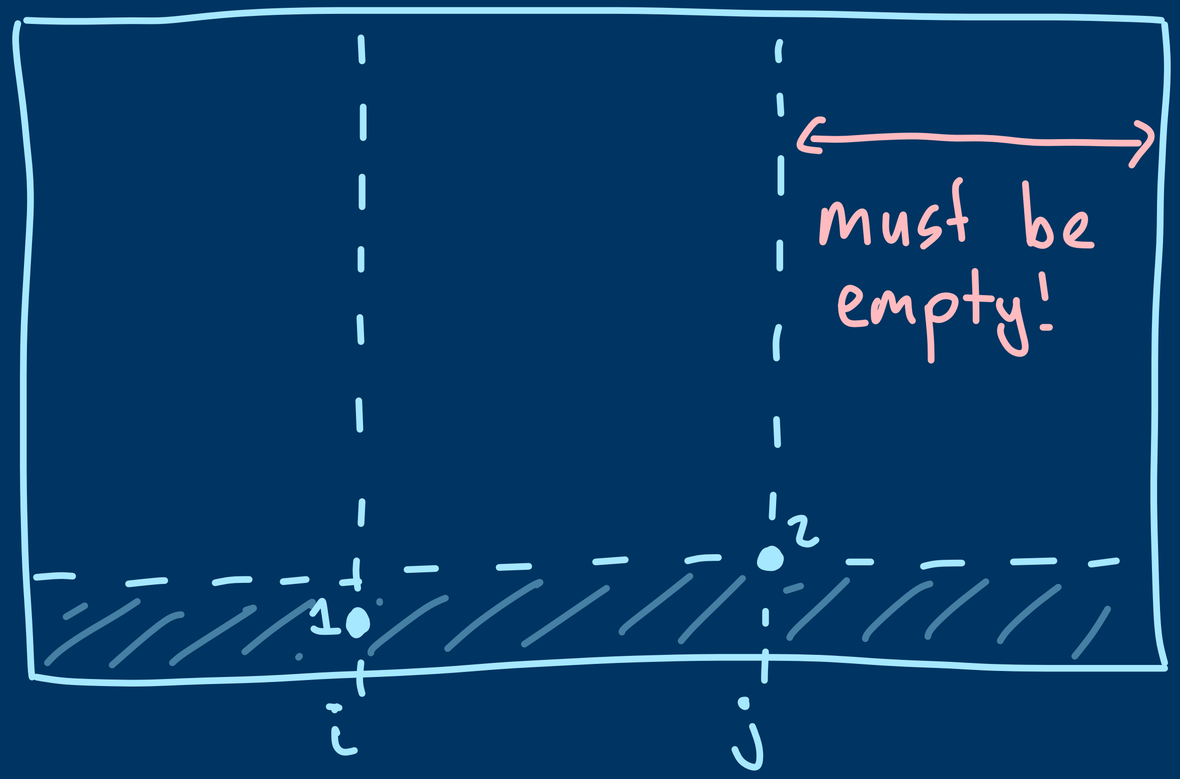
Enumeration Schemes

"The Most Trivial Non-Trivial Example" – $\mathrm{Av}(123)$

Claim 2: $|A_{12}(n,i,j)| = \begin{cases} 0, & j < n \\ |A_1(n-1,i)|, & j = n \end{cases}$



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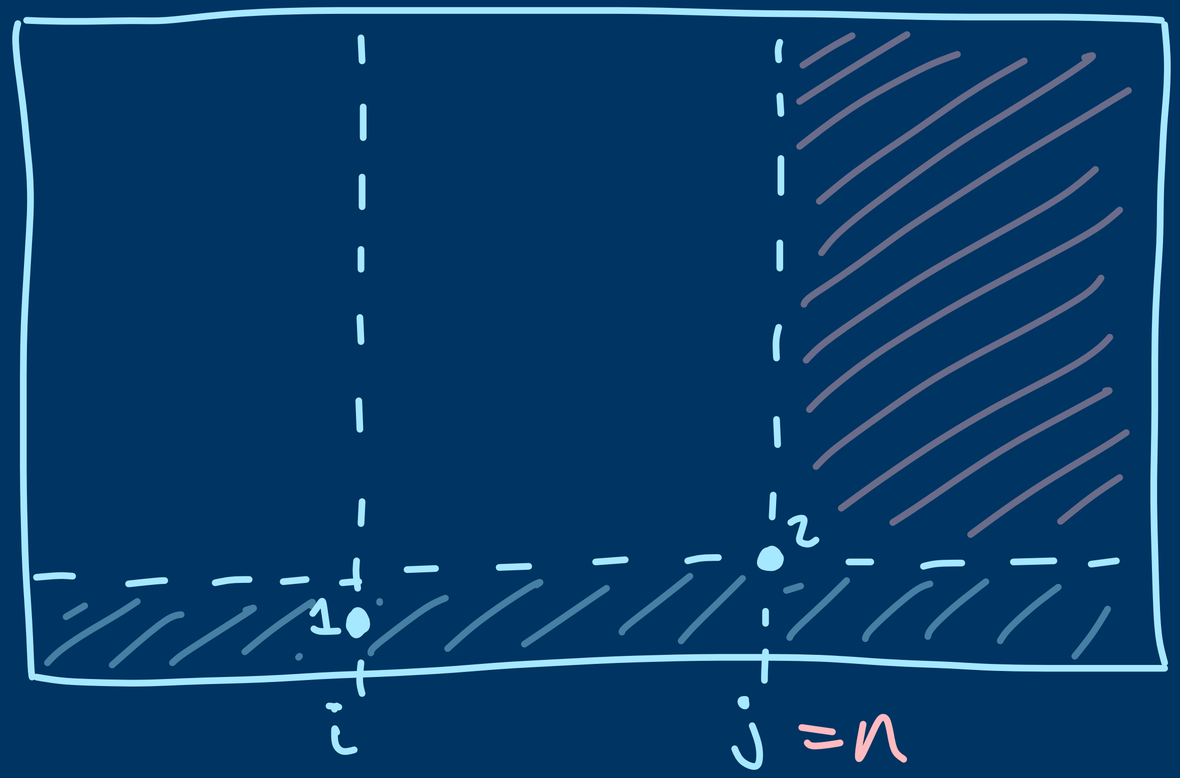

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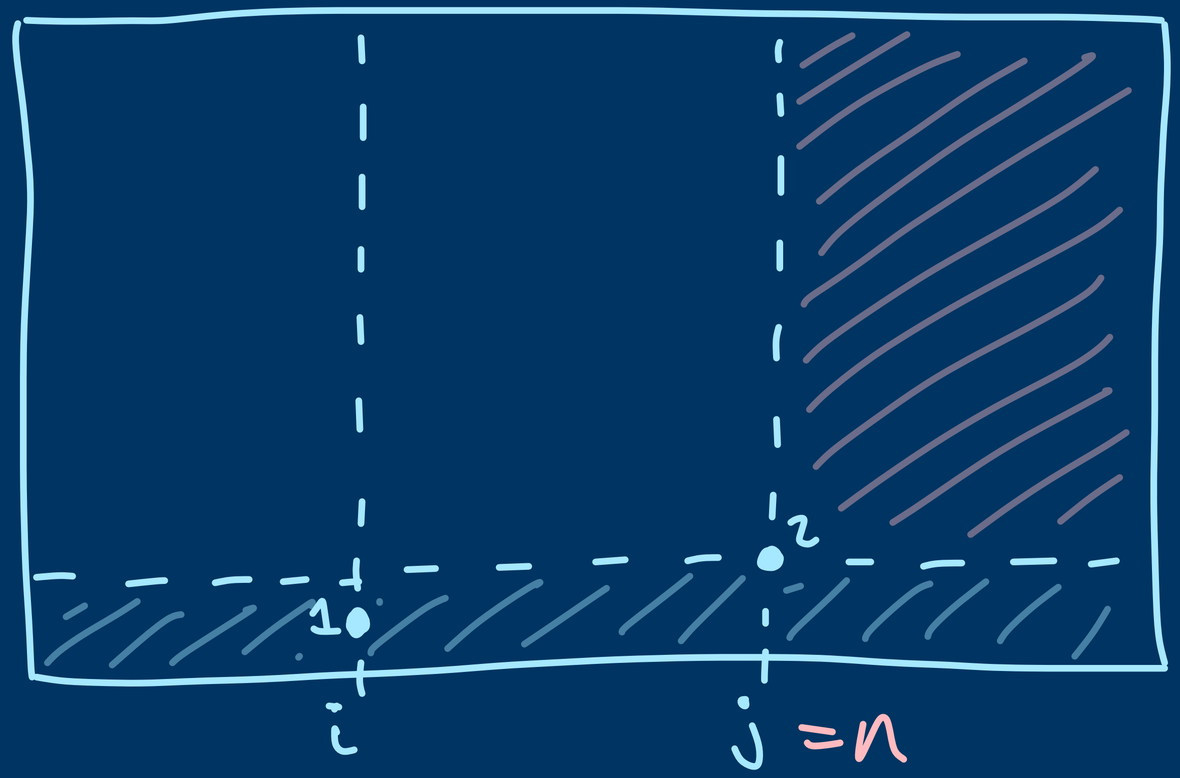

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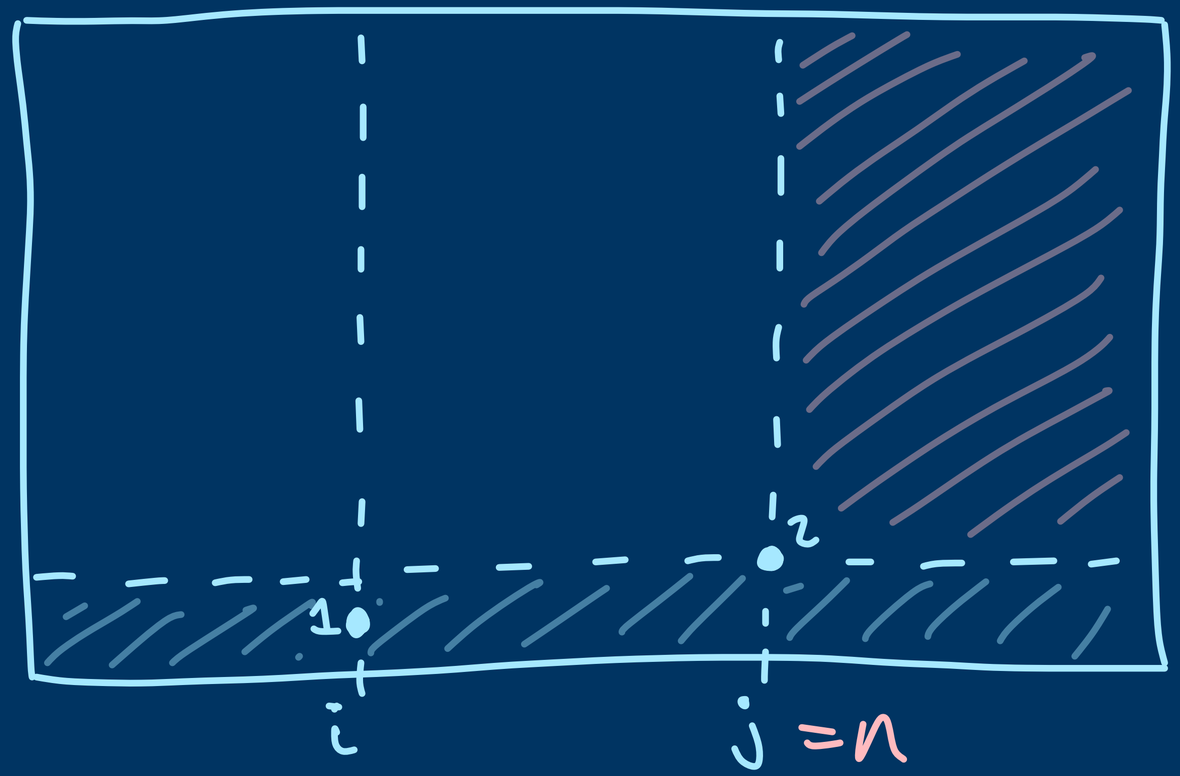

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Enumeration Schemes

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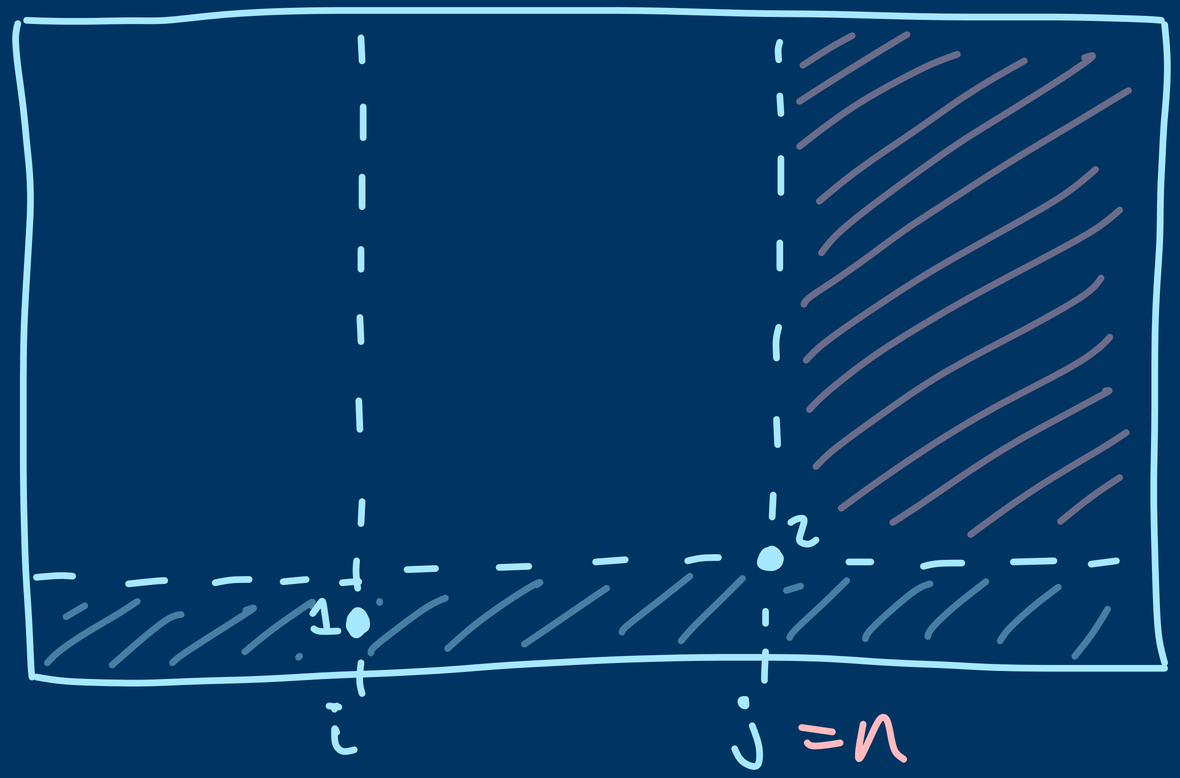

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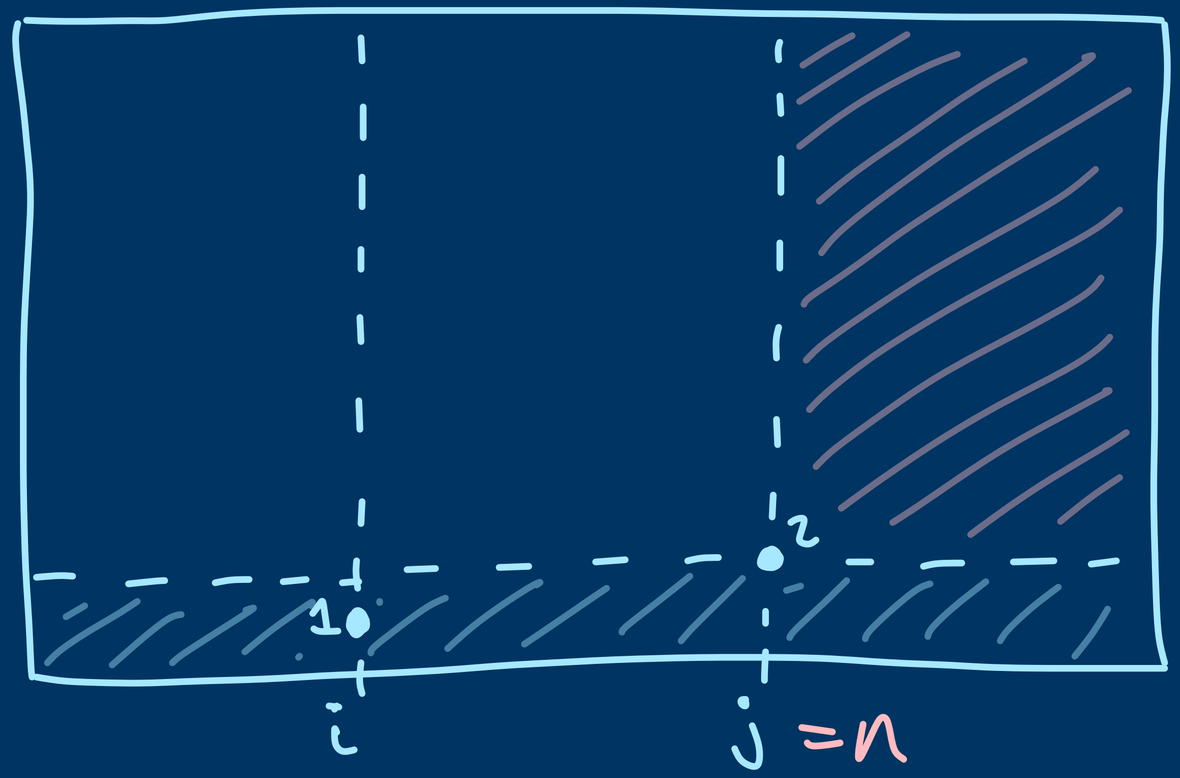

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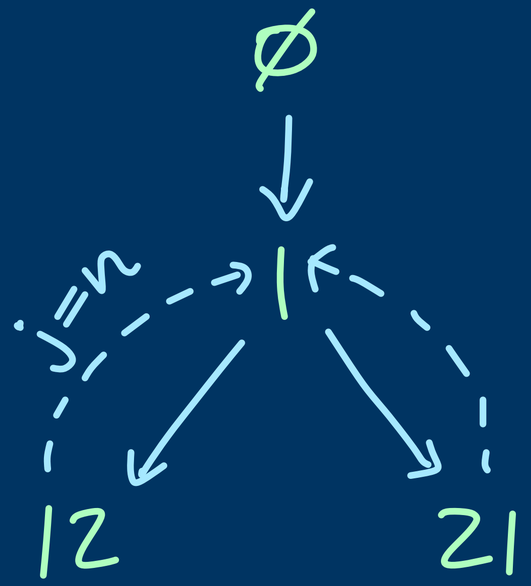
$$A(n) = \bigcup_{i=1}^{n} A_1(n,i)$$

$$A_1(n,i) = \left(\bigcup_{j=1}^{i-1} A_{21}(n,j,i) \right) \cup \left(\bigcup_{j=i+1}^{n} A_{12}(n,i,j) \right)$$

$$A_{21}(n,j,i) \cong A_1(n-1,j)$$

$$A_{12}(n,i,j) \cong \begin{cases} \varnothing & , j < n \\ A_1(n-1,i), & j = n \end{cases}$$



Enumeration Schemes

Big Picture:

- The computer splits the whole set $A(n)$ further and further based on the pattern formed by the bottom entries.

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- At each step it checks if any of the entries are “reversibly deletable”. If so, this branch of the search tree doesn’t need to be split further.
- If all branches finish, we get an enumeration scheme, which gives us a *polynomial-time algorithm* to count the number of permutations of length n, but does not give us the *generating function*.

Enumeration Schemes

Zeilberger's method is:

Enumeration Schemes and, More Importantly, Their Automatic Generation

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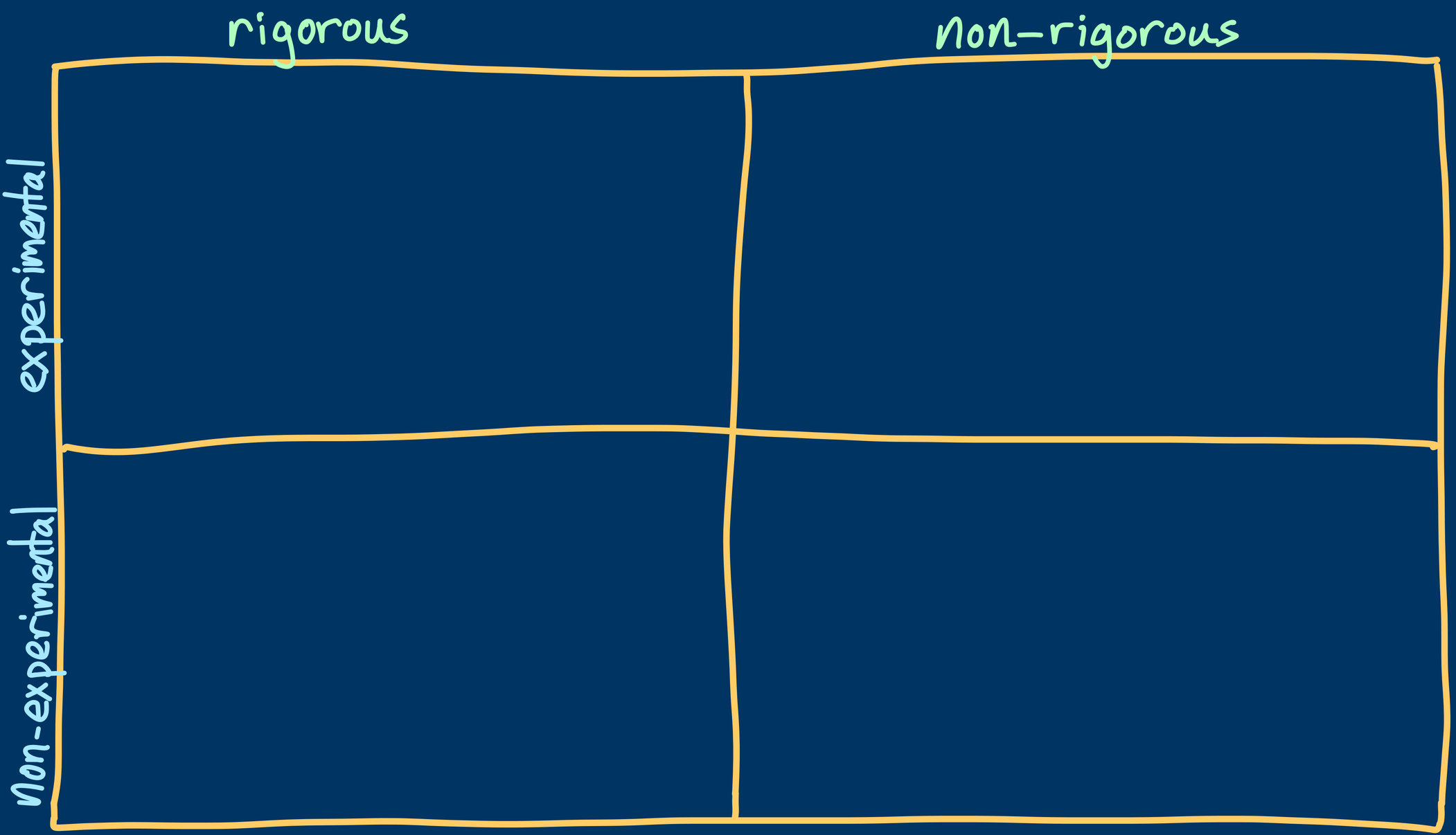
Experimental: when you "hit go", you don't know whether or not it will return an answer

Rigorous: if it does give an answer, it's guaranteed to be correct


rigorous
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experimental
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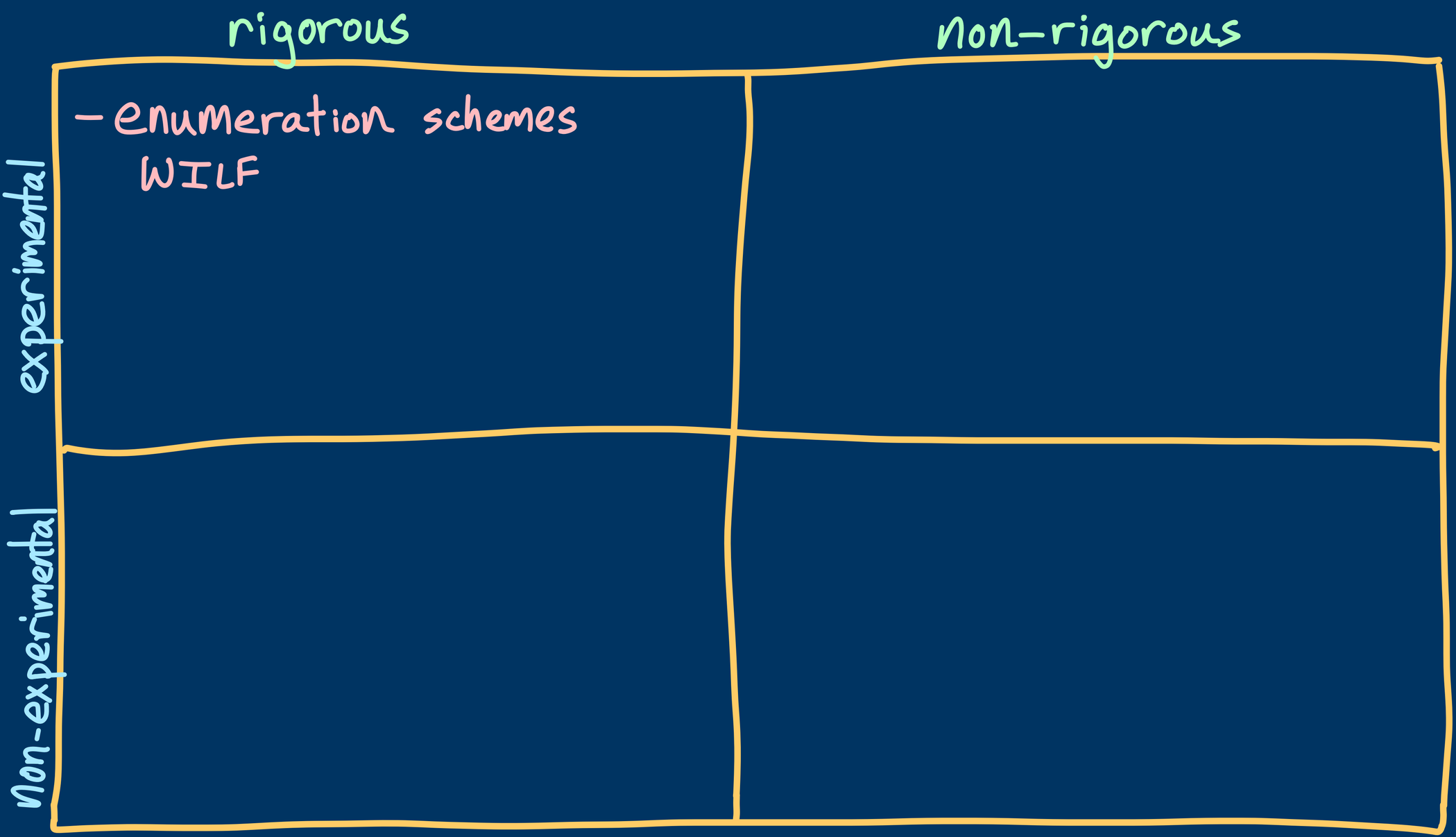
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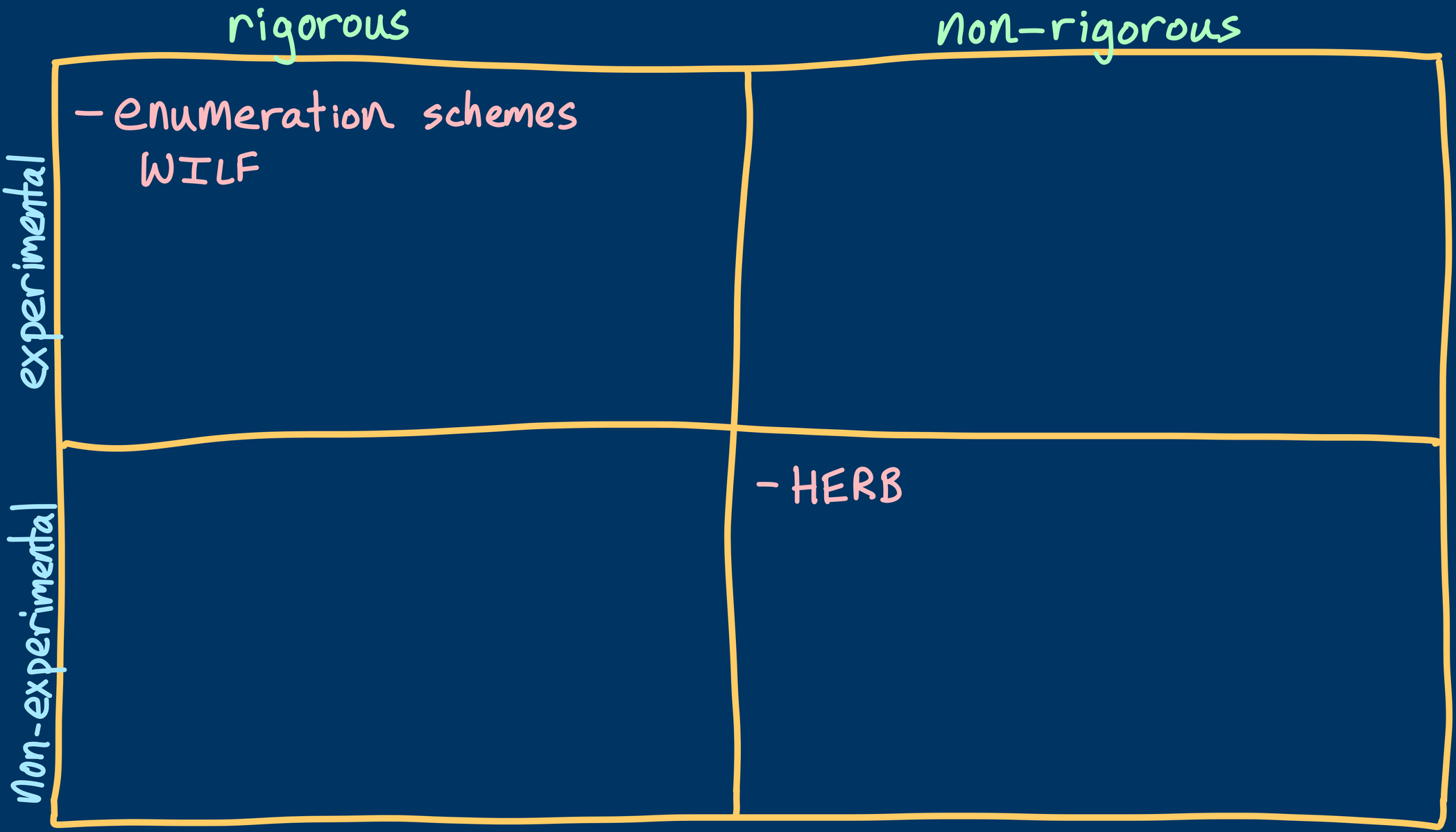
- enumeration schemes
WILF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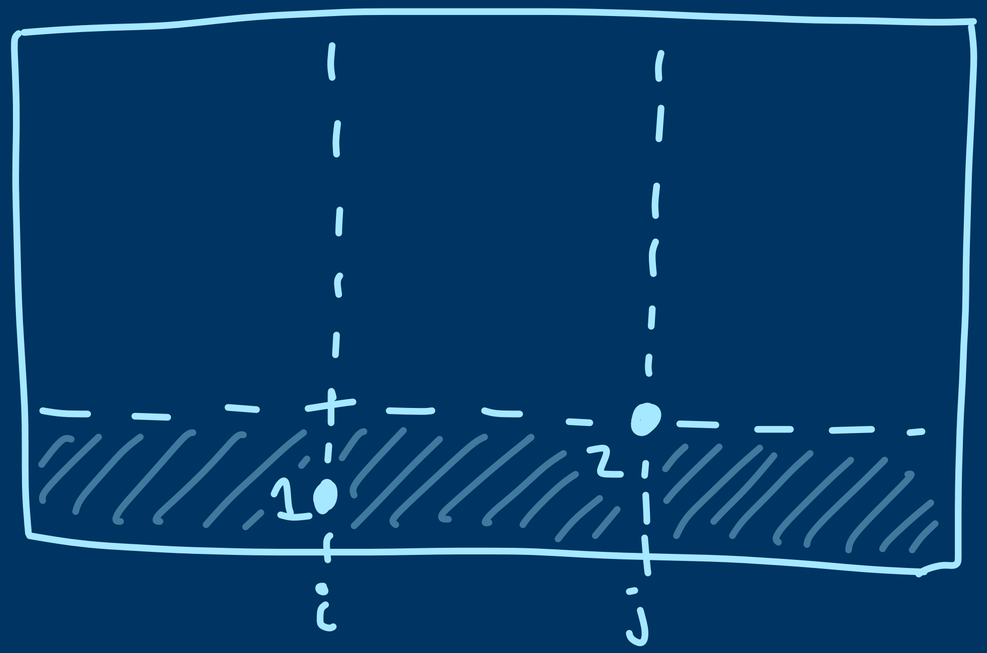
Enumeration Schemes — WILFPLUS

In 2007, Vince Vatter made the method more powerful by increasing the number of situations in which a point can be declared reversibly deletable.

Av(1342,1432)



Enumeration Schemes for Restricted Permutations

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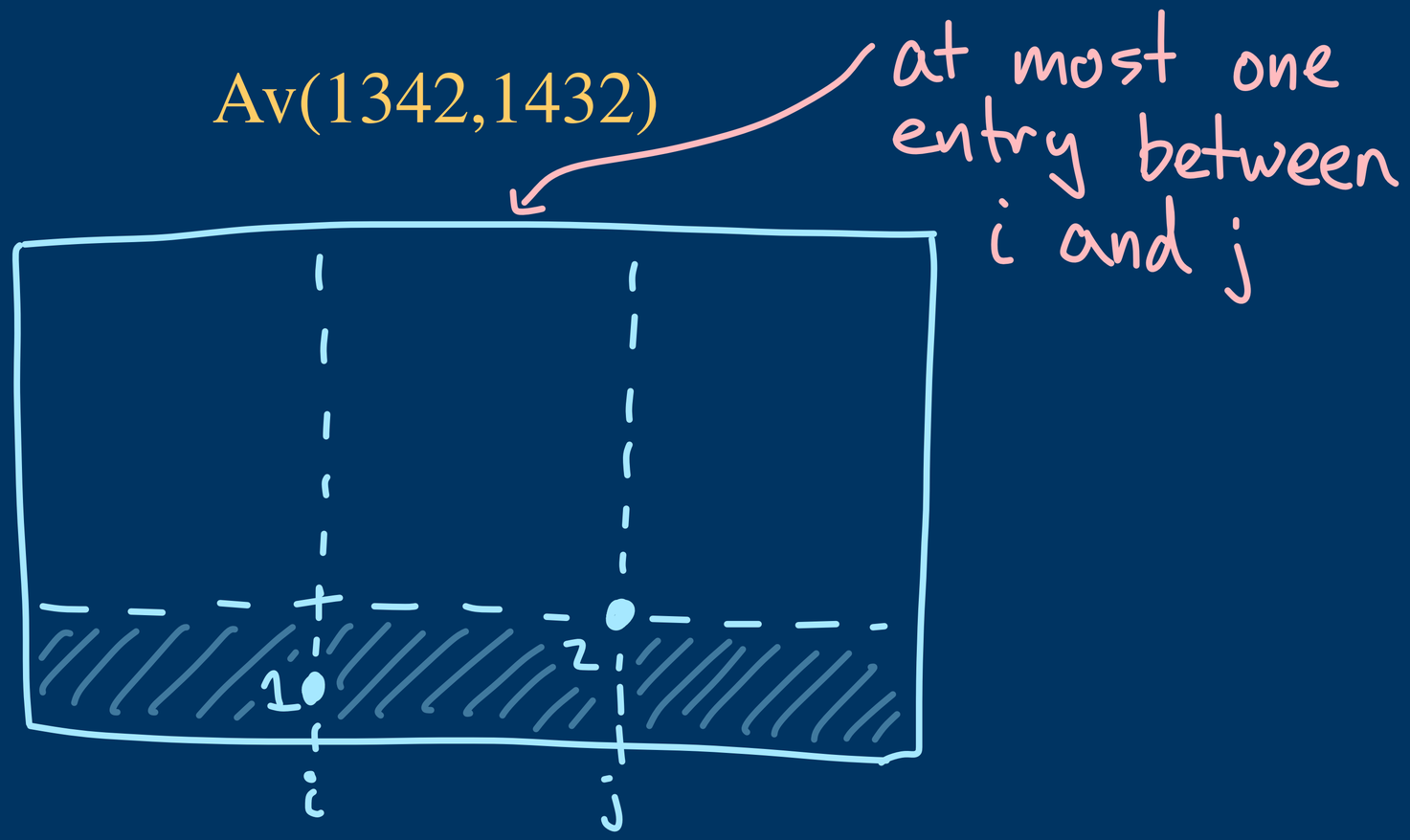

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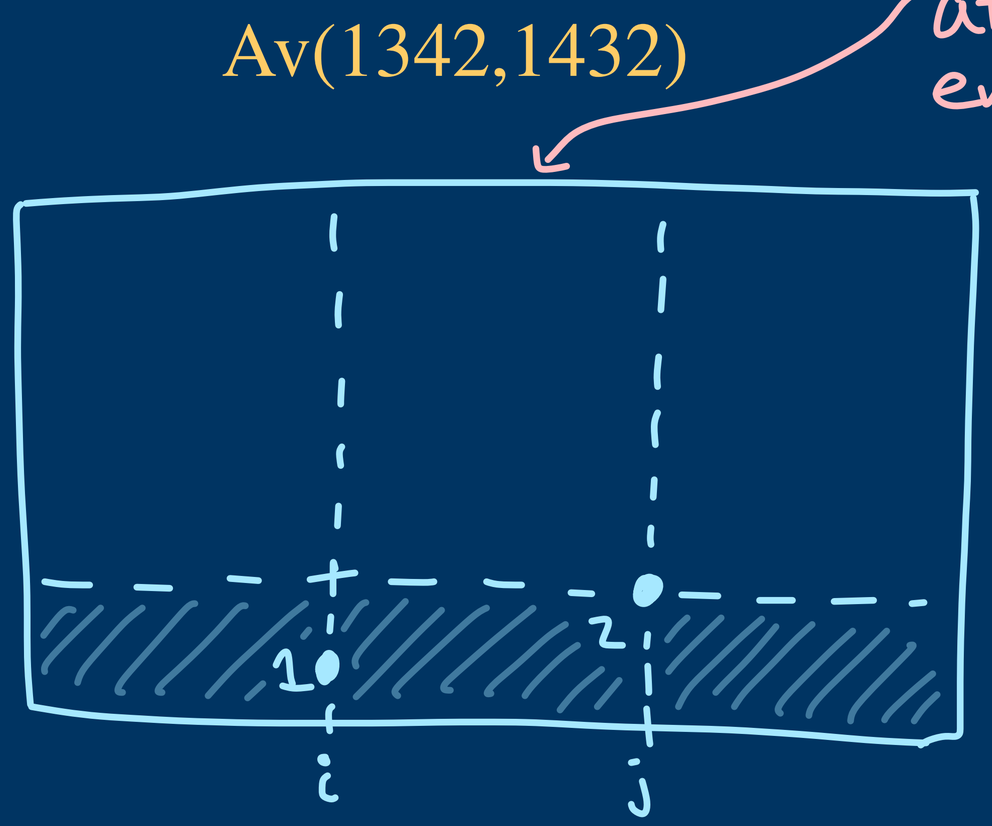
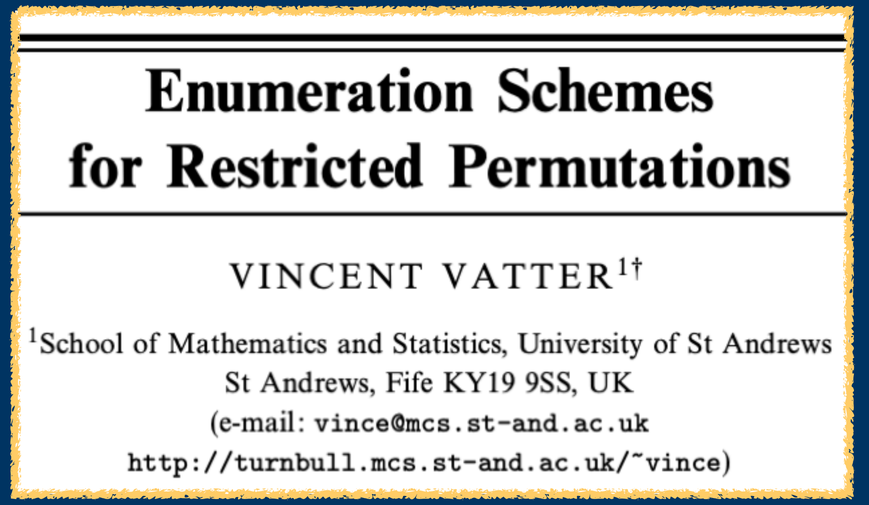


Knowing this: the entry $\pi(j)$ can be deleted.



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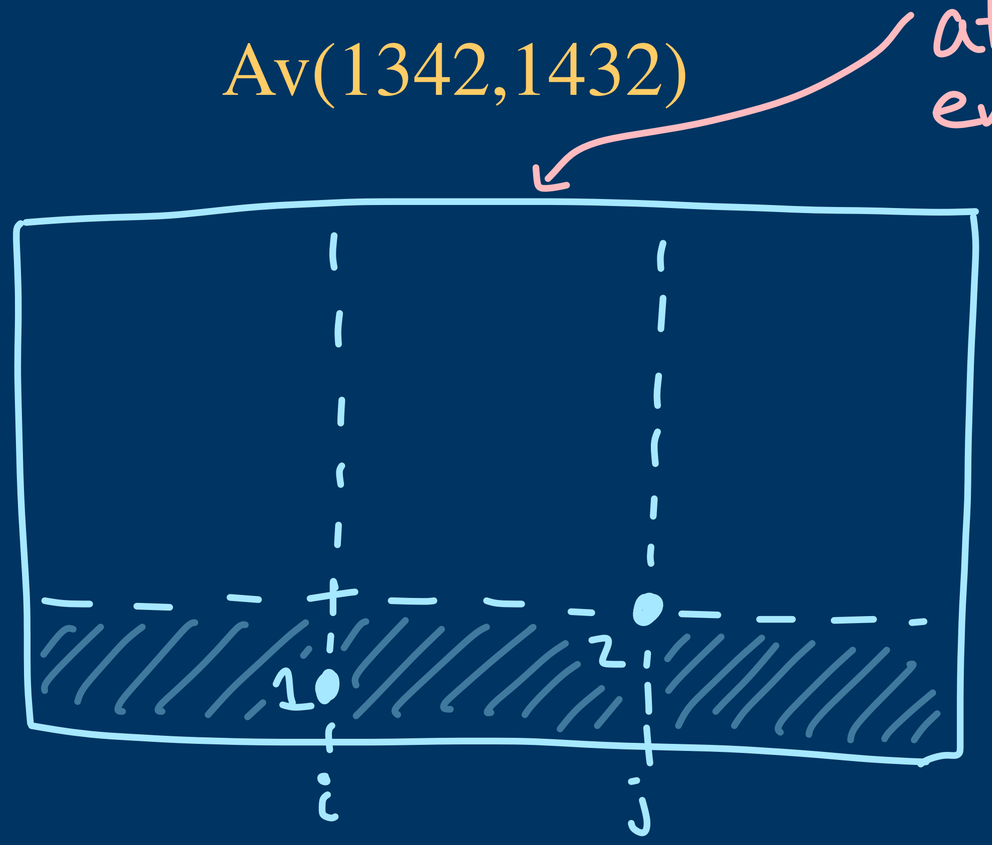

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Zeilberger's "logical reasoning" won't notice this.

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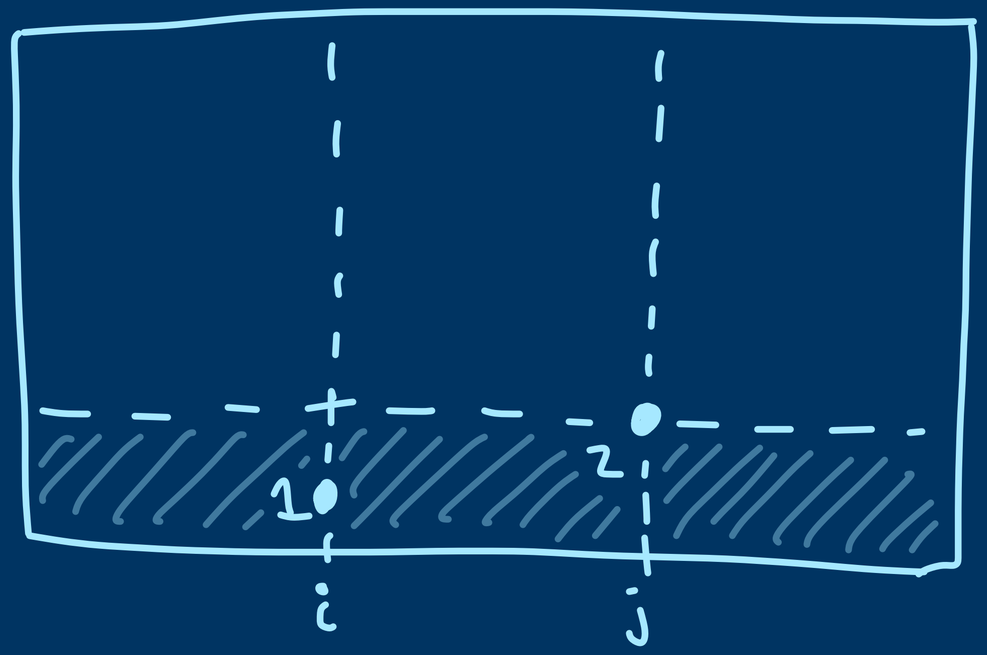

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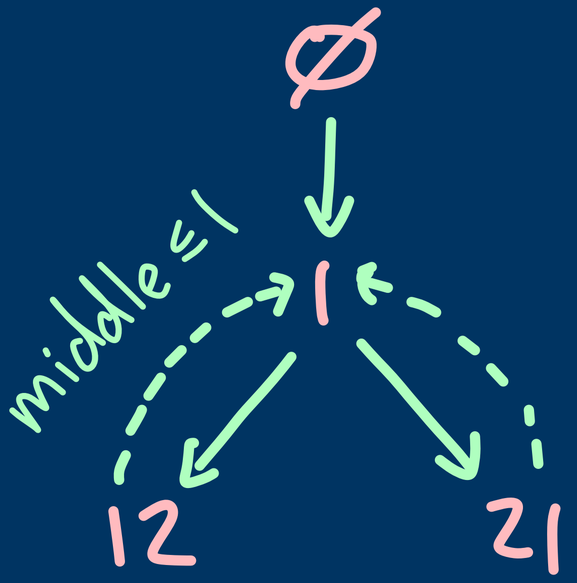

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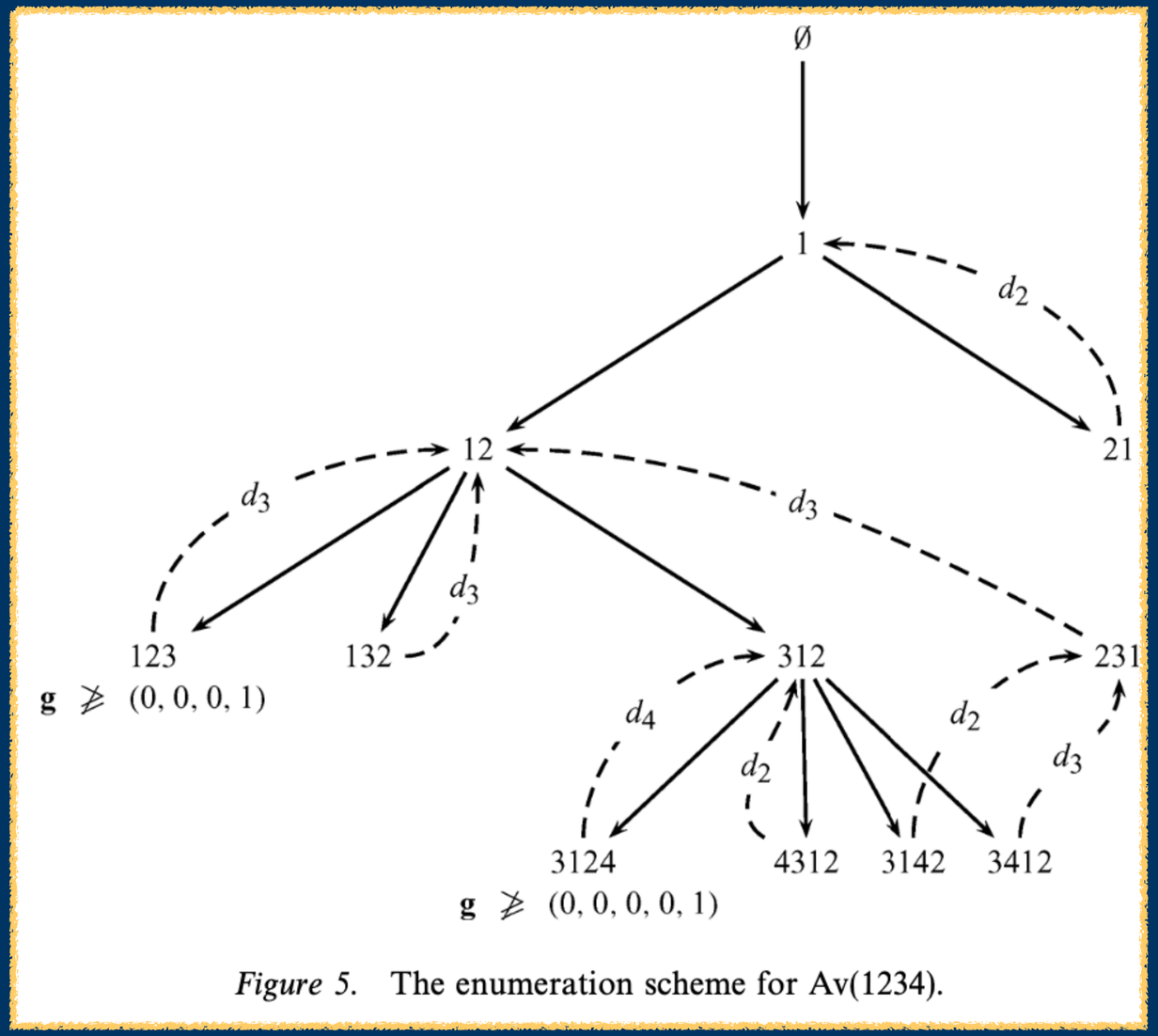

Figure 5. The enumeration scheme for Av(1234).

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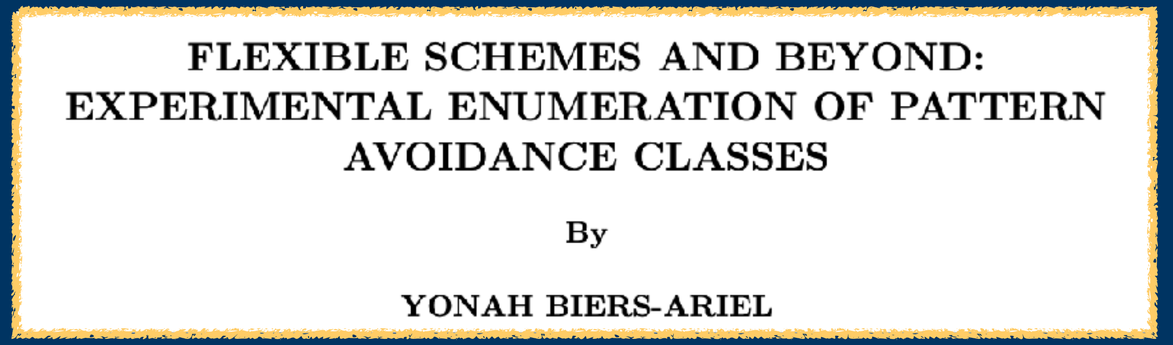
Enumeration Schemes — Flexible Schemes

Z: When checking if a point is reversibly deletable, can take into account whether a gap between two entries must be empty.

can do only a few simple classes



FLEXIBLE SCHEMES AND BEYOND:
EXPERIMENTAL ENUMERATION OF PATTERN AVOIDANCE CLASSES

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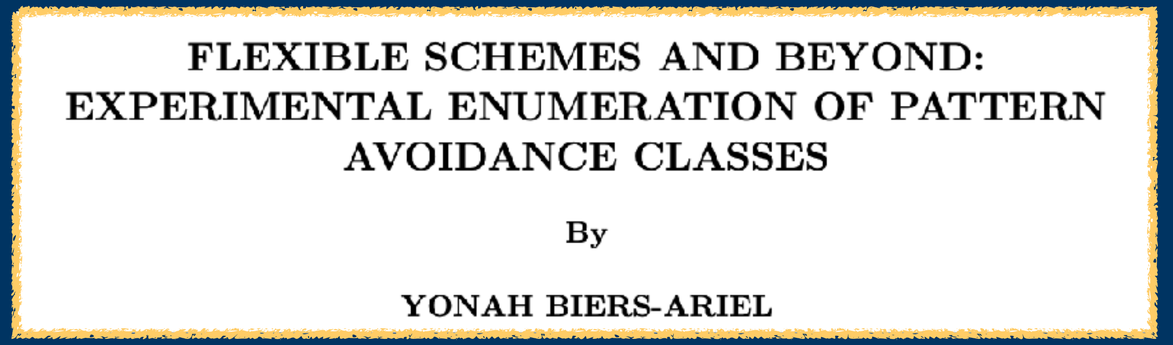
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V: Can take into account when a gap is constrained to have a finite number of entries (and more complicated similar constraints)

can do more classes

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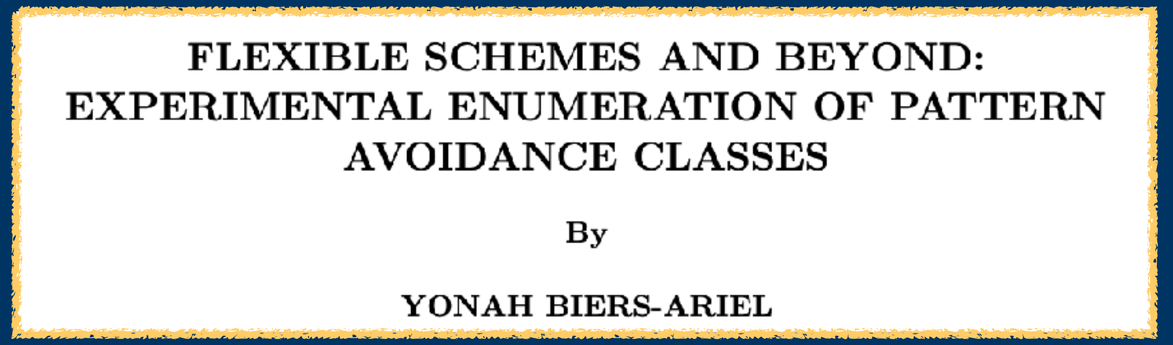
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B-A: Sometimes if a gap is constrained to a finite number of possibilities, there could be one entry deletable for some of these possibilities, and a *different entry* deletable for the other possibilities.

can do even more classes

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can do only a few simple classes

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By

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Pat length[1]	Sym Classes[2]	Ins. Enc.	ES	FS	New with FS
[3]	2	0	2	2	0
[4]	7	0	2	2	0
[5]	23	0	2	2	0
[3], [3]	5	5	5	5	0
[4], [4]	56	13	33	44	9
[4], [5]	434	30	112	173	59

V: Ca ned to have a finite number of entri ts)

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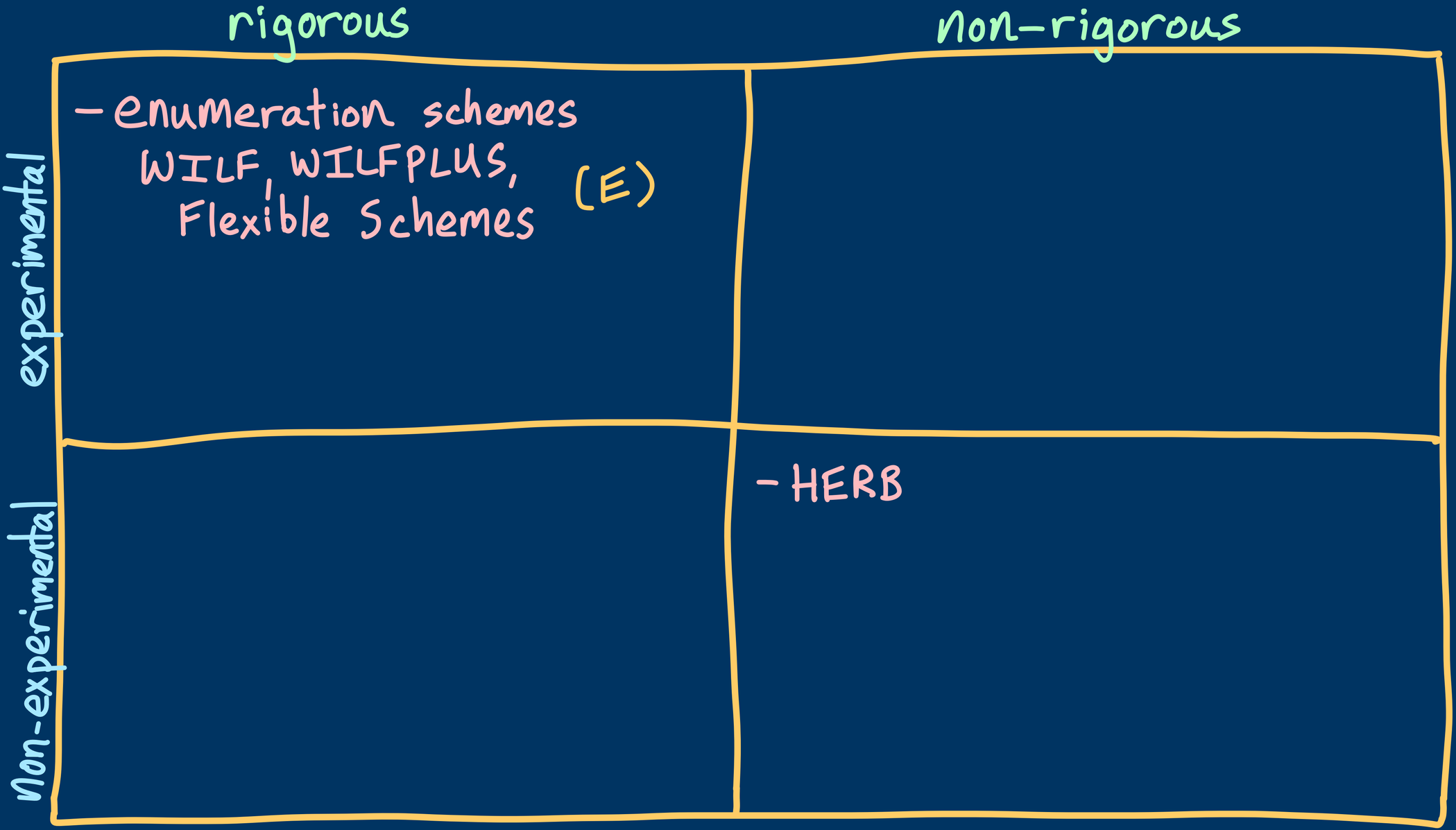
rigorous
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WILF, WILFPLUS,
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(E)
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rigorous

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Enumeration schemes for vincular patterns

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exp

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exp

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Refining enumeration schemes to count according to permutation statistics

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Pennsylvania , U.S.A.

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exp

non-experimental

Enumeration Schemes for Permutations Avoiding Barred Patterns

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Department of Mathematics and Computer Science
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Lara.Pudwell@valpo.edu

Refining enumeration schemes to count according to permutation statistics

Andrew M. Baxter

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Generating Trees

- A “generating tree” for a set of permutations is a way of rigorously representing its structure. It describes where new maximum entries can be inserted into permutations so that they remain in the set.

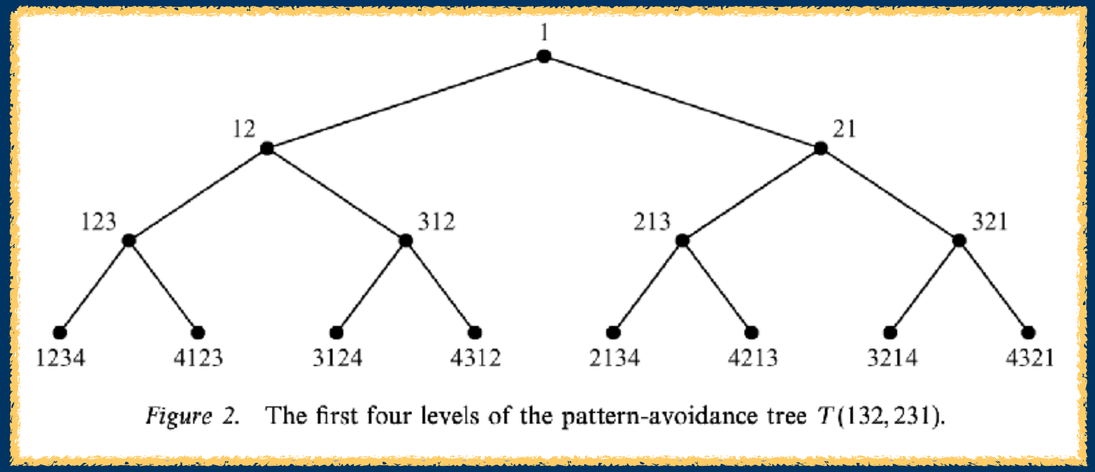

Figure 2. The first four levels of the pattern-avoidance tree $T(132, 231)$.

(Vatter 2007)

Generating Trees

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- 1995/1996: West uses generating trees to enumerate several permutation classes.

- 2006: Vatter categorizes the permutation classes that have finitely labeled generating trees and writes the Maple package FINLABEL to enumerate them automatically.

Generating Trees

- 1998 — present: ECO Method

  Exports the idea of generating trees to other combinatorial objects and uses them to do many things: enumeration, generating functions, exhaustively generating all objects in a fast way, ...

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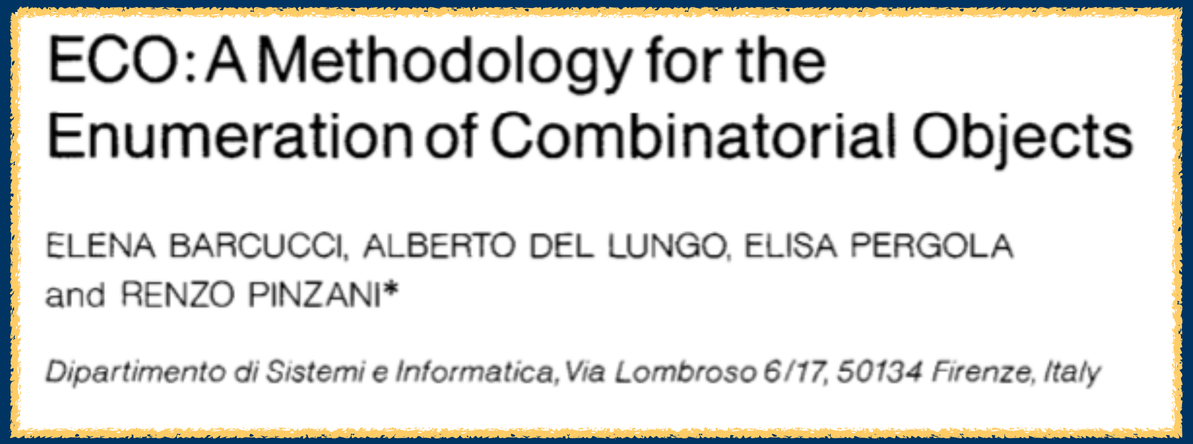
- 1998 — present: ECO Method

  Exports the idea of generating trees to other combinatorial objects and uses them to do many things: enumeration, generating functions, exhaustively generating all objects in a fast way, ...



ECO: A Methodology for the Enumeration of Combinatorial Objects

ELENA BARCUCCI, ALBERTO DEL LUNGO, ELISA PERGOLA and RENZO PINZANI*

Dipartimento di Sistemi e Informatica, Via Lombroso 6/17, 50134 Firenze, Italy

Generating Trees

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Some applications arising from the interactions between the theory of Catalan-like numbers and the ECO method*

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Integer Partitions in Discrete Dynamical Models and ECO Method*

Le Manh Ha[1] and Phan Thi Ha Duong[2]

[1] *College of Education, Hue University, 34 Le Loi, Hue, Vietnam*

[2] *Institute of Mathematics, 18 Hoang Quoc Viet Road, 10307 Hanoi, Vietnam*

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ECO Method and the Exhaustive Generation of Convex Polyominoes

Alberto Del Lungo, Andrea Frosini, and Simone Rinaldi

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ECO-generation for some restricted classes of compositions

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Generating involutions, derangements, and relatives by ECO

Vincent Vajnovszki

LE2I – UMR CNRS, Université de Bourgogne, B.P. 47 870, 21078 DIJON-Cedex France.
Email: vvajnov@u-bourgogne.fr

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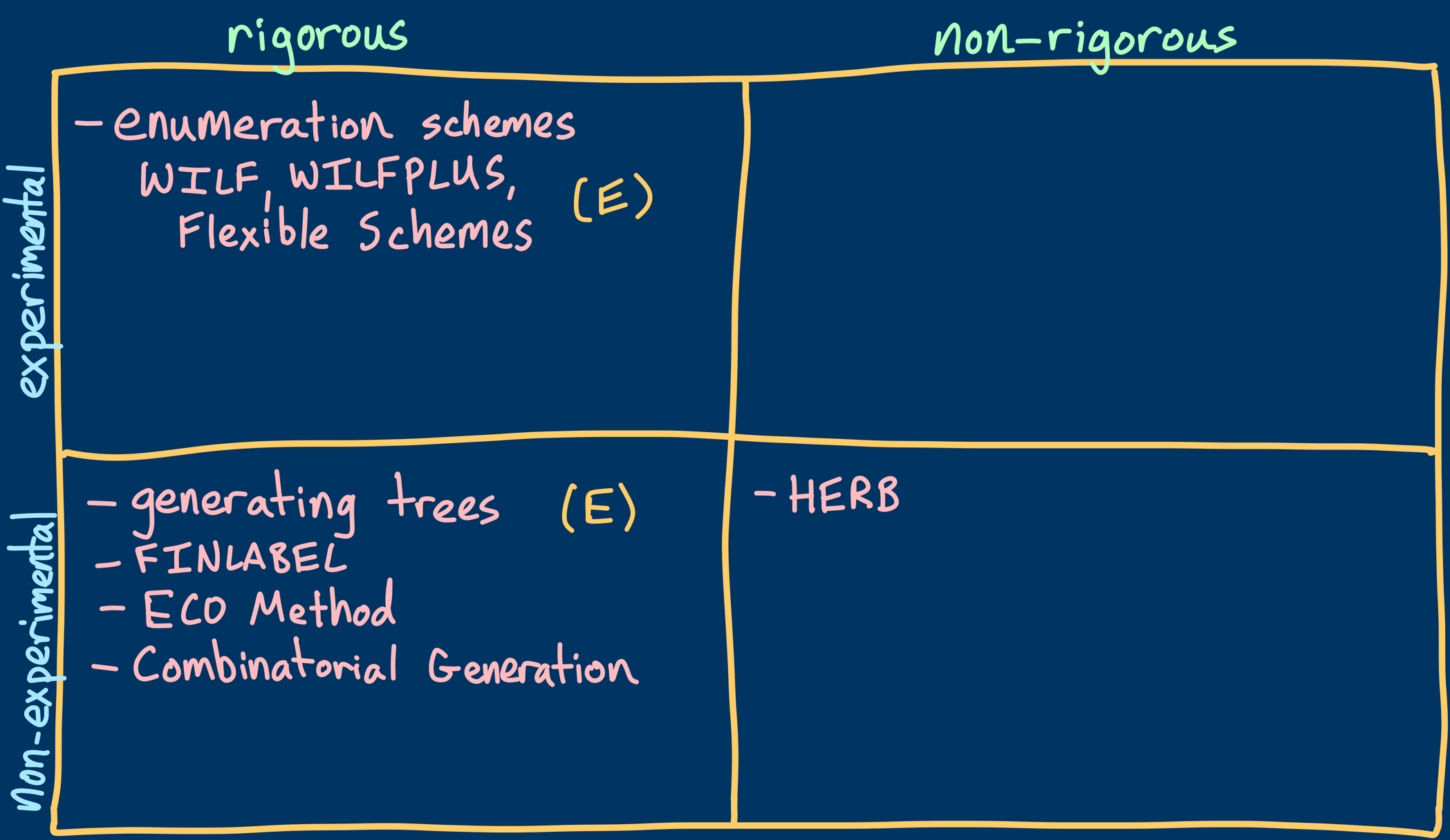
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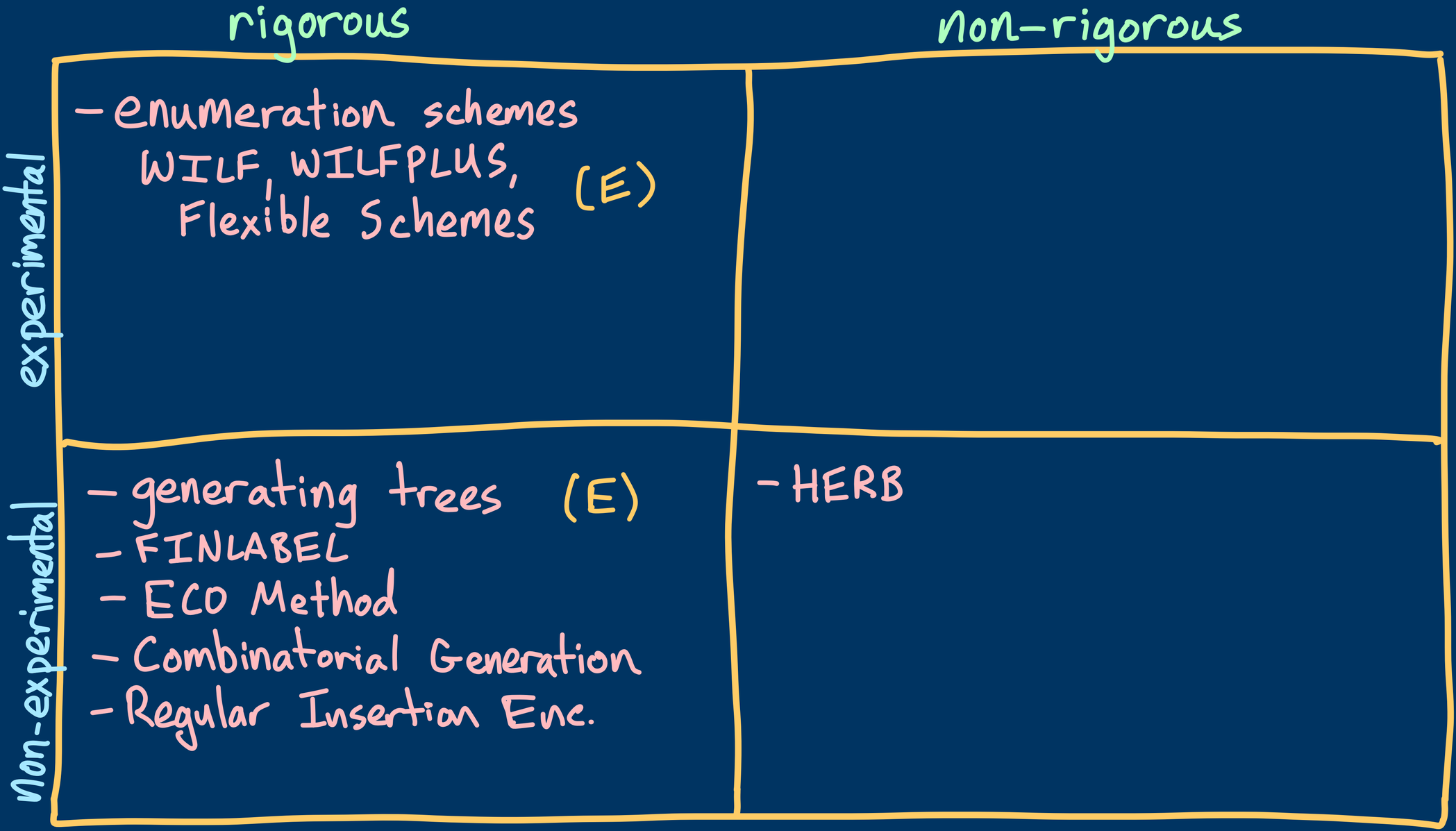
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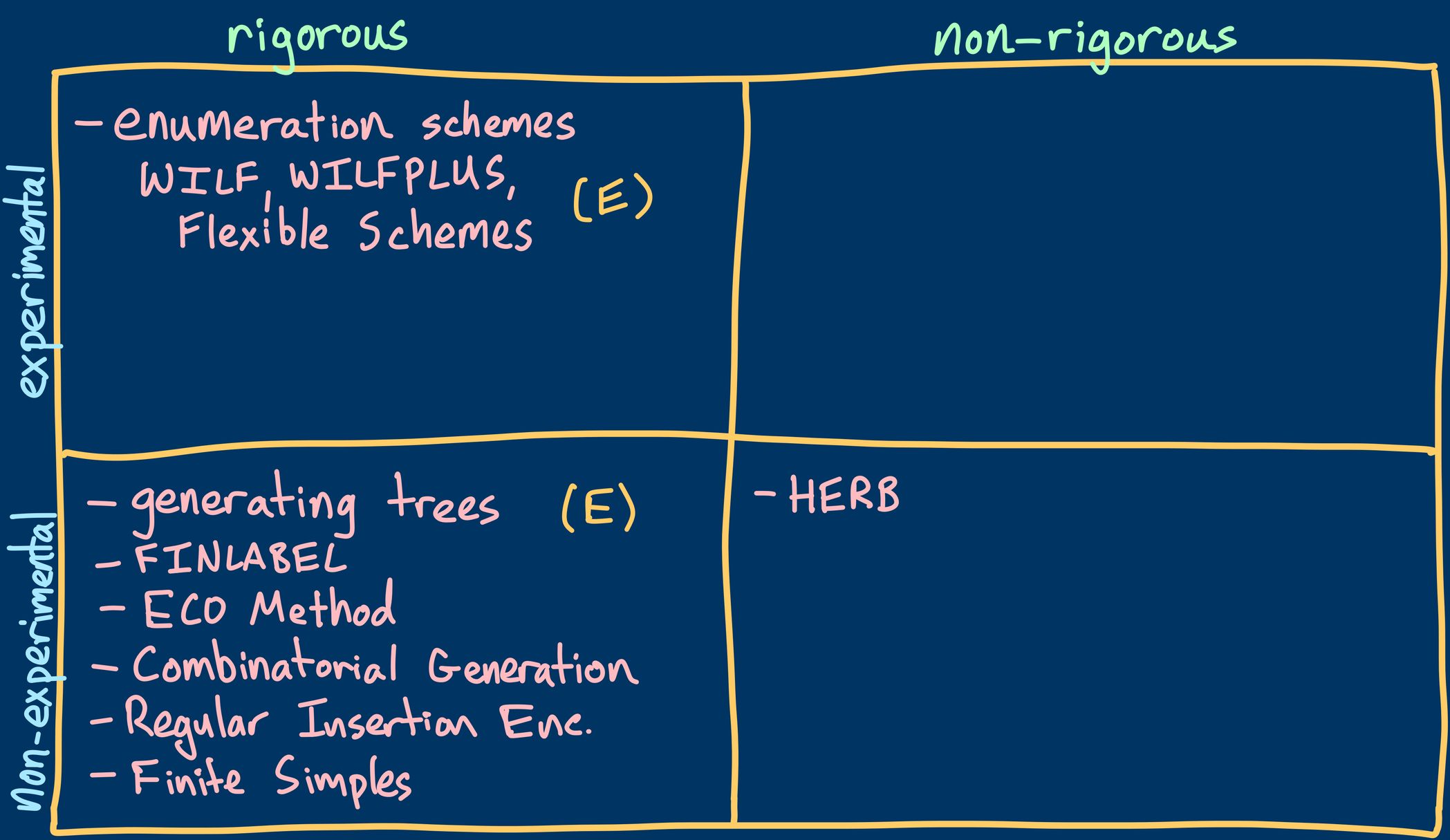
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Struct

Most of the methods described so far:

"expand a particular structure tree and hope it ends up being finite"

MATHEMATICS OF COMPUTATION
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AUTOMATIC DISCOVERY OF STRUCTURAL RULES OF PERMUTATION CLASSES

CHRISTIAN BEAN, BJARKI GUDMUNDSSON, AND HENNING ULFARSSON

Struct is a software package that takes a permutation class as input and searches for a *set cover* that decomposes it into simpler disjoint parts.

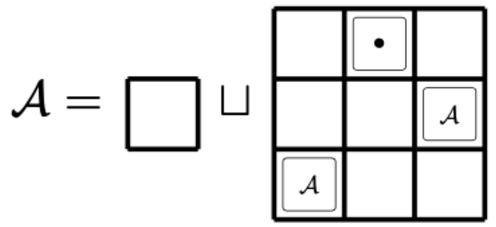

FIGURE 1. The structure of Av(231)

Struct

$\mathcal{G}_4 = \mathrm{Av}(321,1324)$

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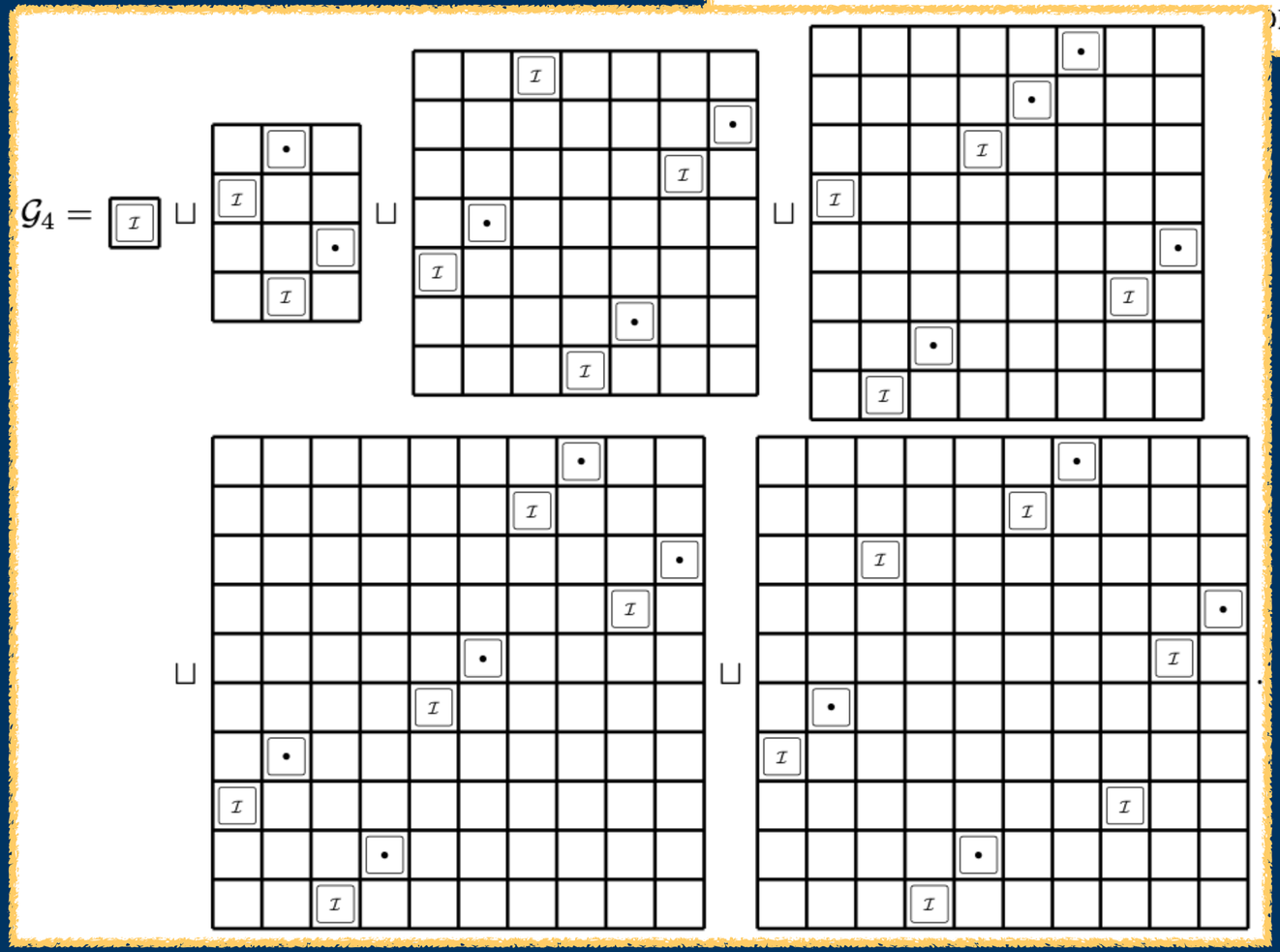
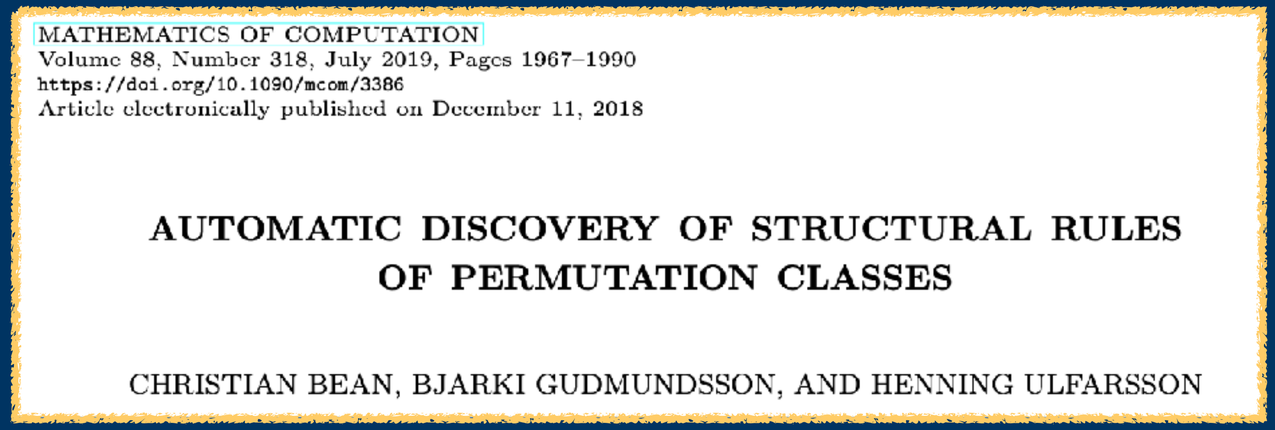


Struct

Method:

- Construct a big list of grids that make subsets of the input class.



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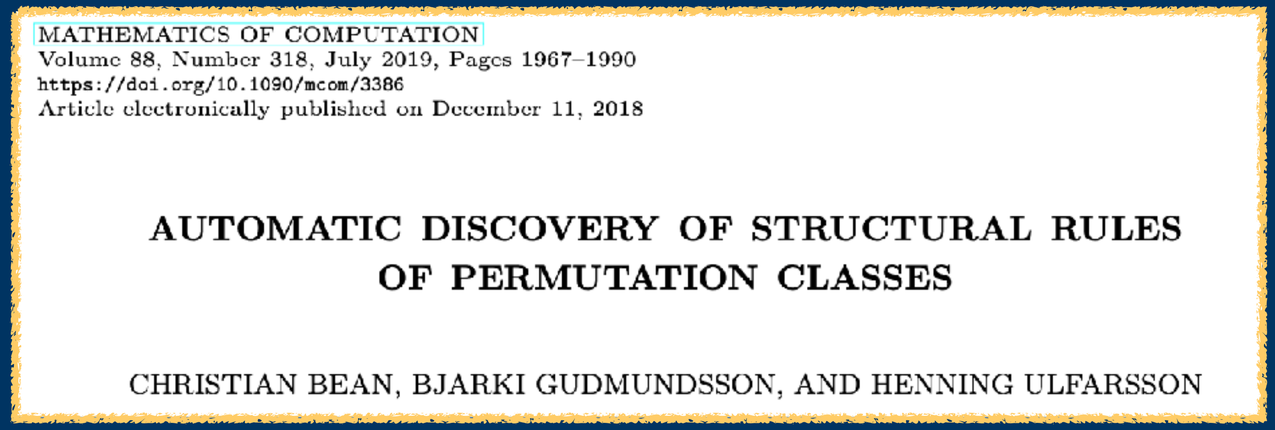
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- Set up an integer linear programming problem to pick a subset of grids that forms a set cover (each permutation in the class gives one constraint).

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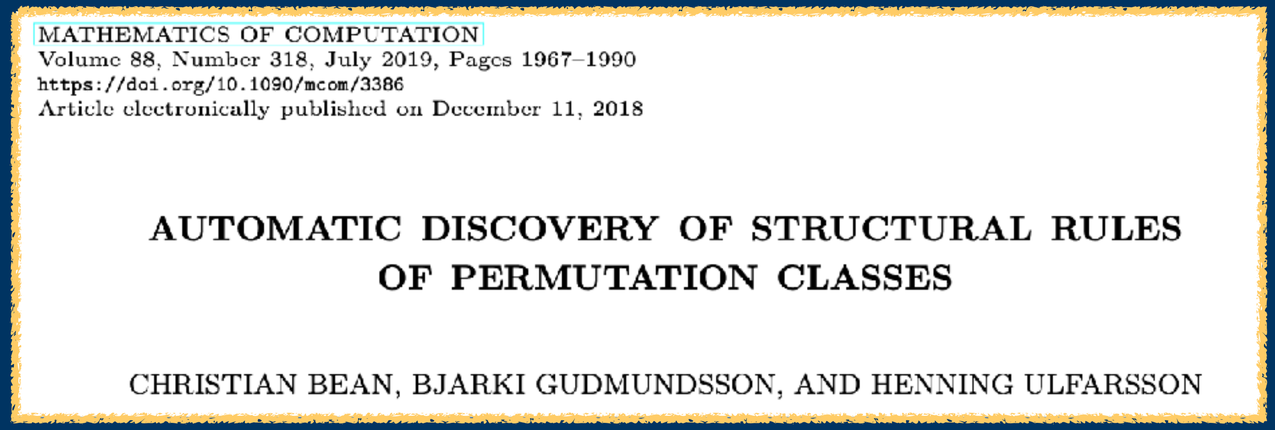
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- Set up an integer linear programming problem to pick a subset of grids that forms a set cover (each permutation in the class gives one constraint).
- Feed it into an ILP solver like Gurobi and wait patiently for a solution.

	rigorous	non-rigorous
experimental	- enumeration schemes WILF, WILFPLUS, Flexible Schemes (E)	- Struct - BiSC
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Combinatorial Exploration

At the end of the Struct paper, the authors discuss some classes that Struct can’t do along with a possible future approach.

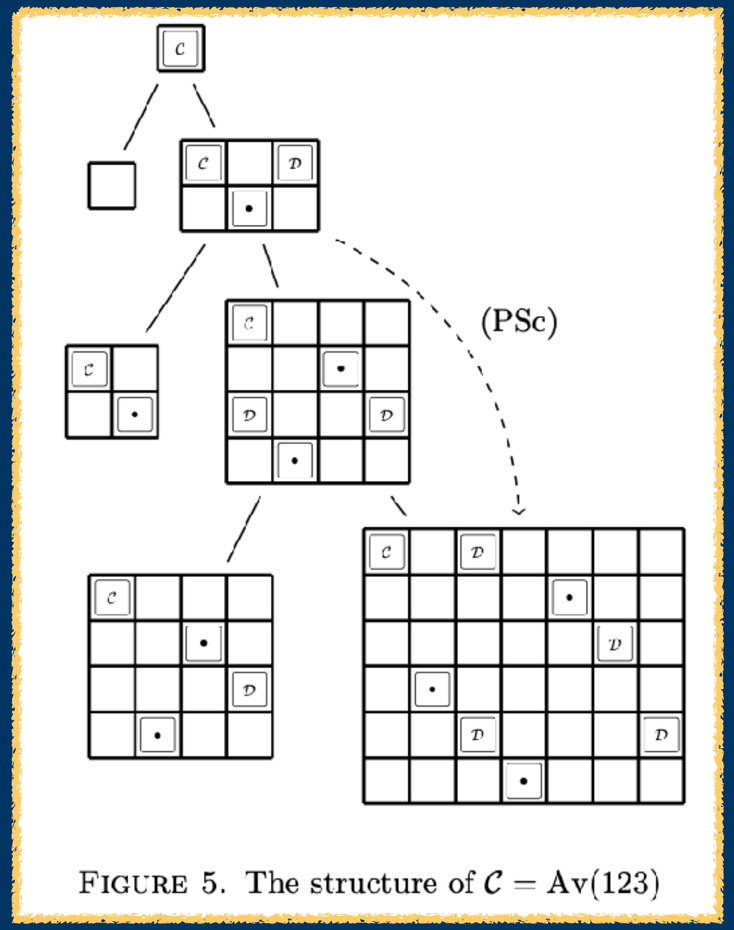


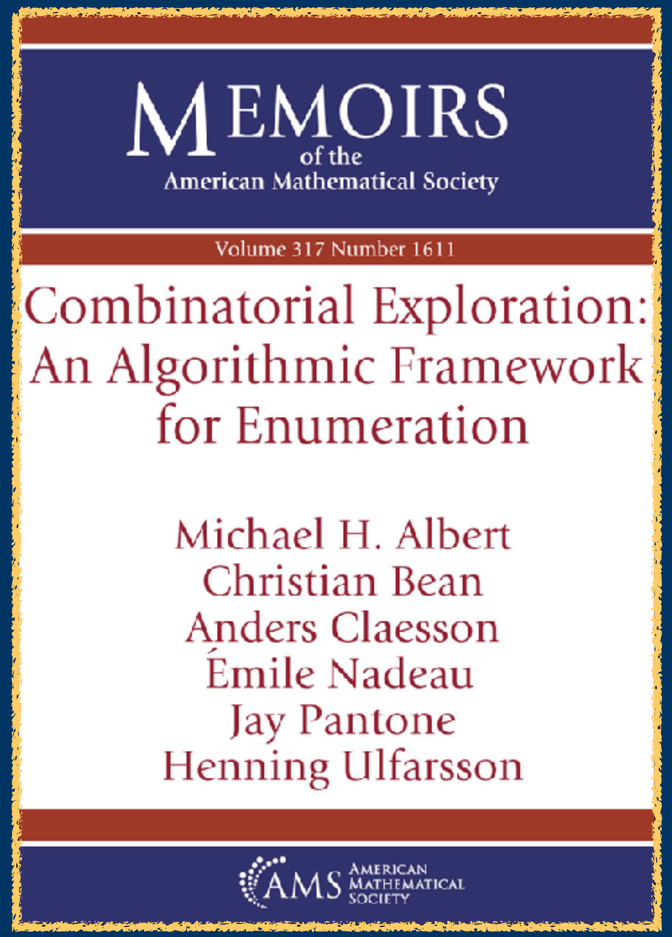
FIGURE 5. The structure of $\mathcal{C} = \mathrm{Av}(123)$

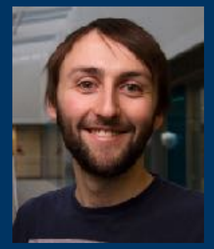
“proof tree”

Combinatorial Exploration

I started talking with Henning Ulfarsson and Christian Bean at PP 2016.

Many years later…





Combinatorial Exploration

Key insights:

1. Instead of expanding one particular structure tree and hoping it ends up being finite: produce a bunch of independent “rules” that relate a parent set to child sets, and hope that some subset of these rules can be assembled into a tree

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Combinatorial Exploration

Key insights:

1. Instead of expanding one particular structure tree and hoping it ends up being finite: produce a bunch of independent “rules” that relate a parent set to child sets, and hope that some subset of these rules can be assembled into a tree
2. We need a much more efficient way to represent sets of permutations.
3. If (1) and (2) are done correctly, then the result can still be fully rigorous.

C

$C = Av(132)$

C^+ = nonempty perms

empty or not

point placement

row/col separation

factor

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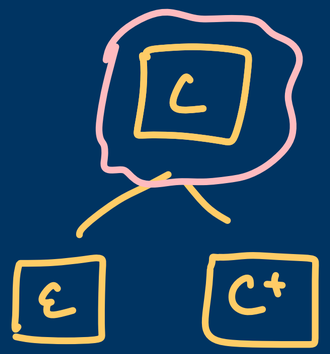

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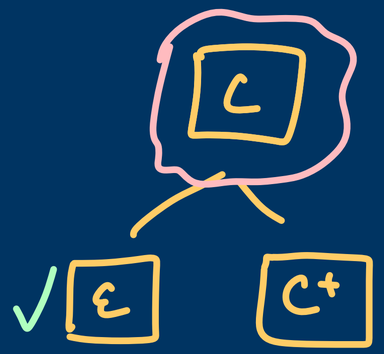

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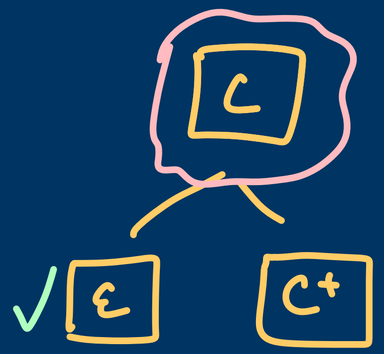

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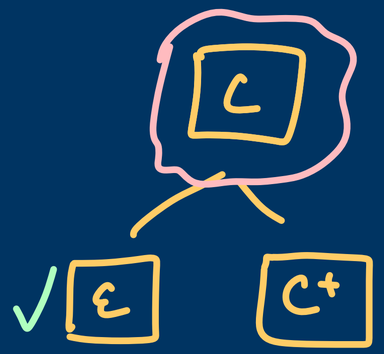

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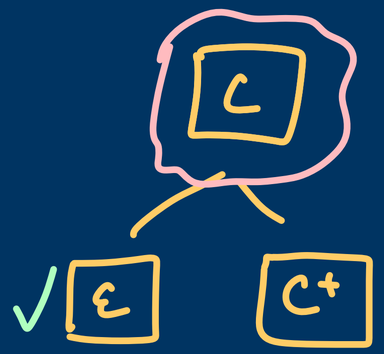

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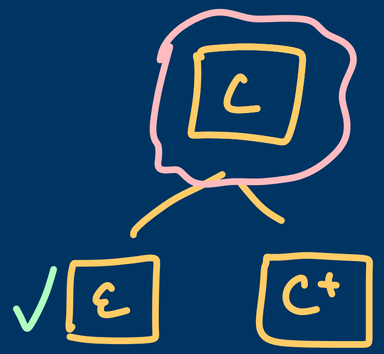

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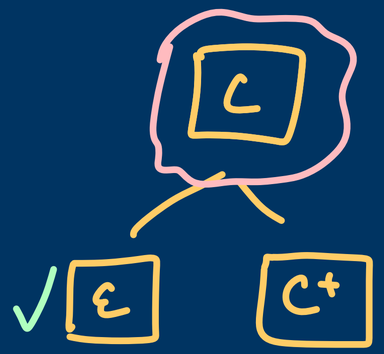

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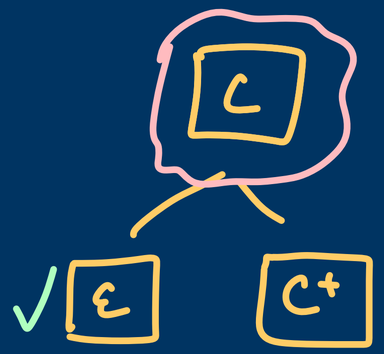

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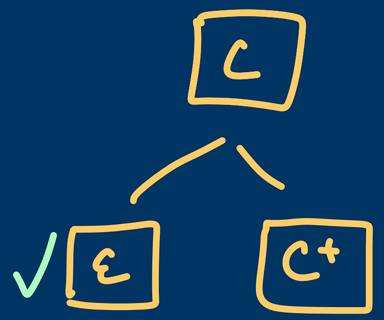

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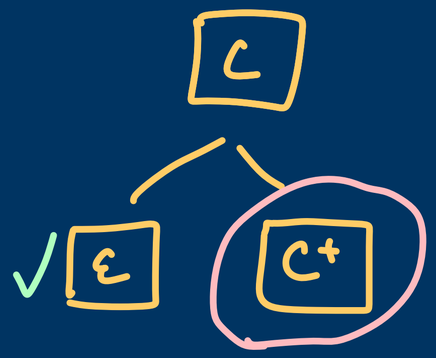

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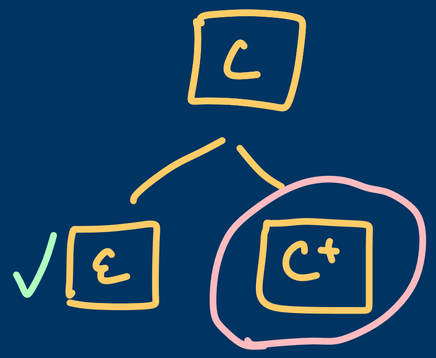

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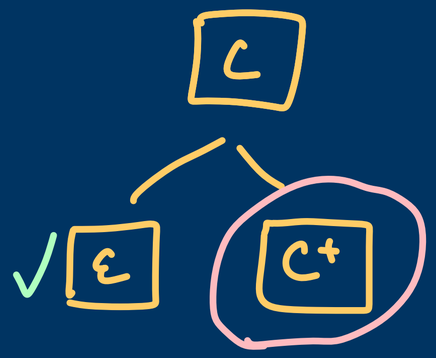

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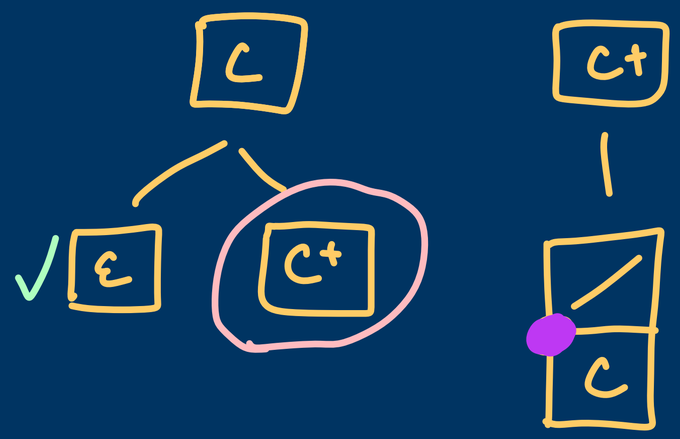

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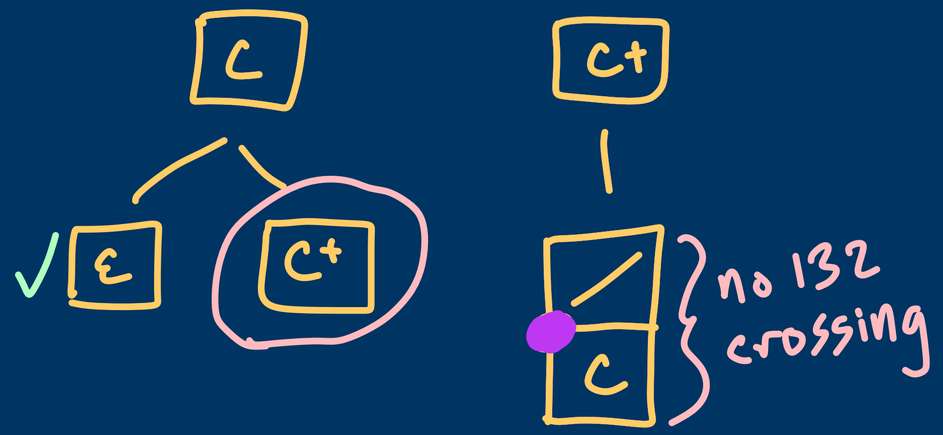

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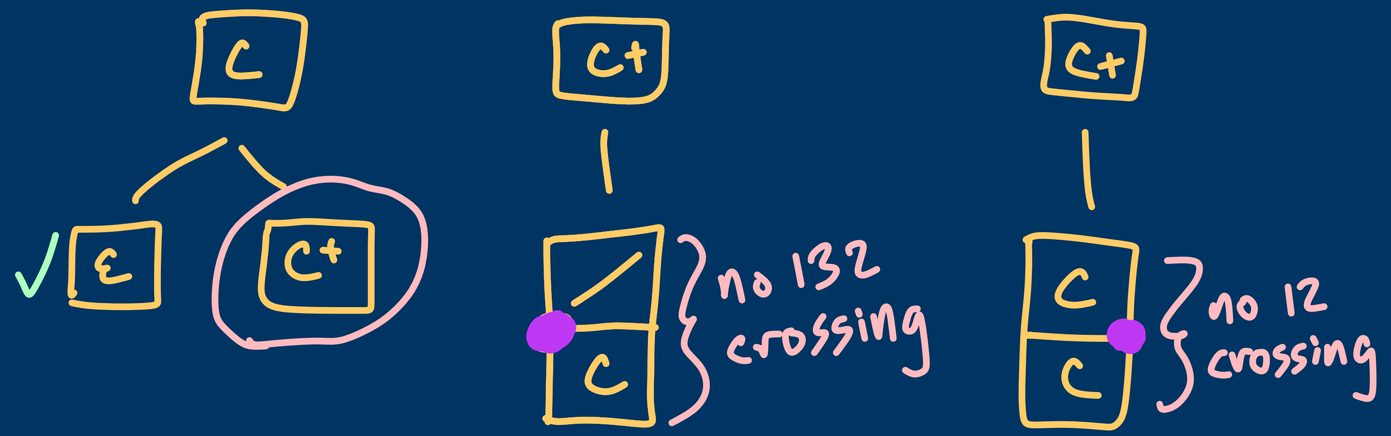

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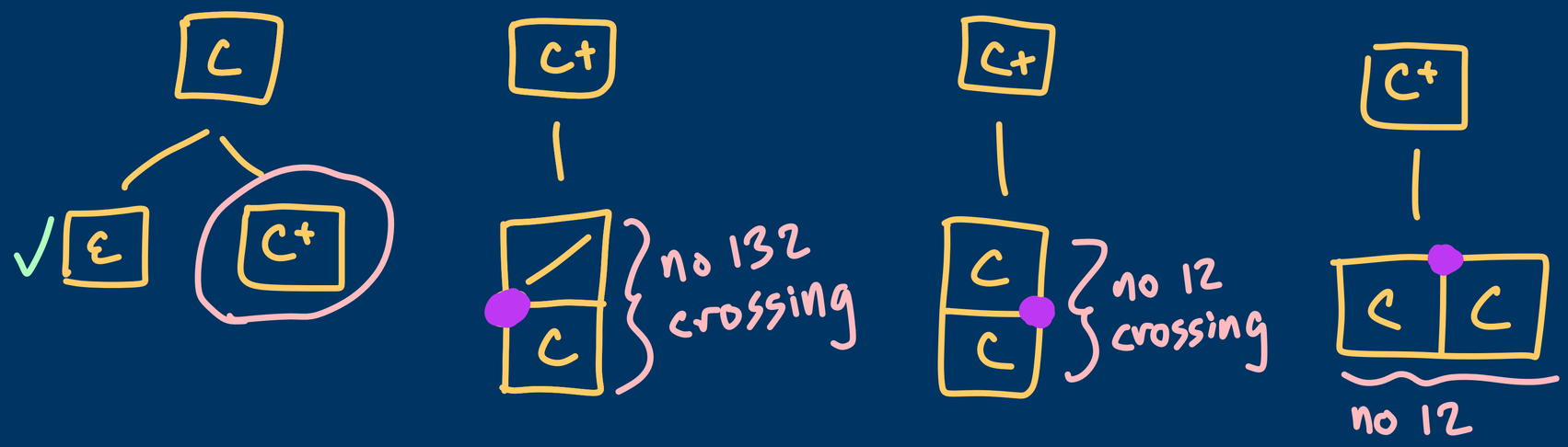

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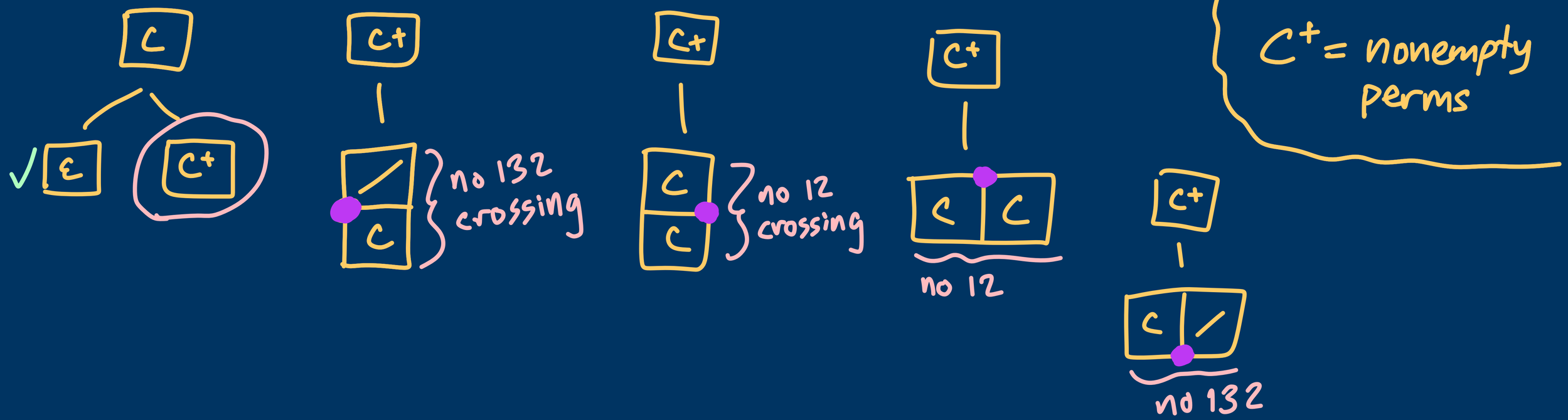

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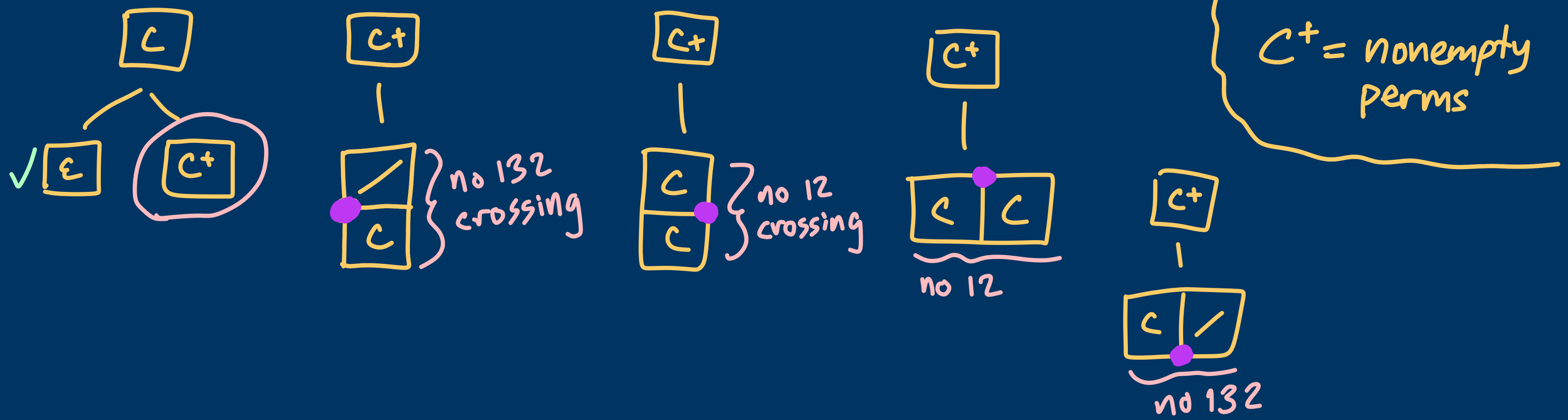

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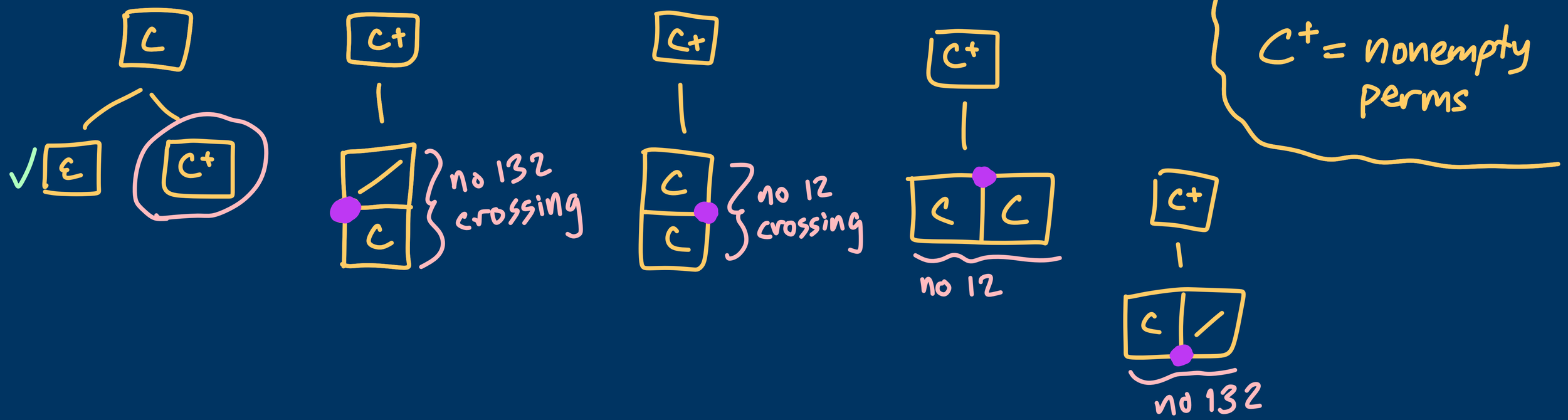

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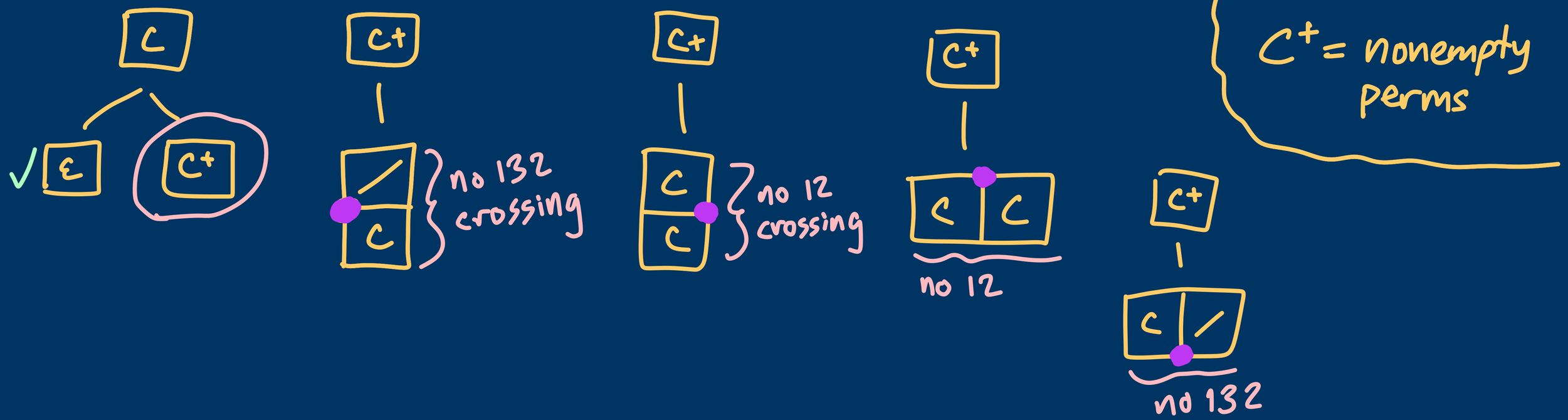

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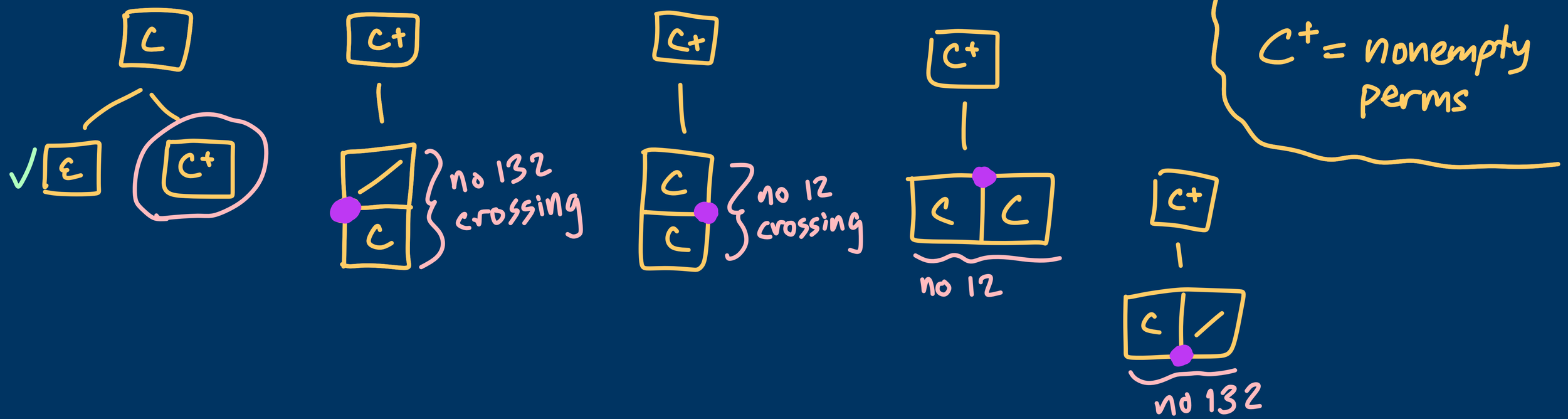

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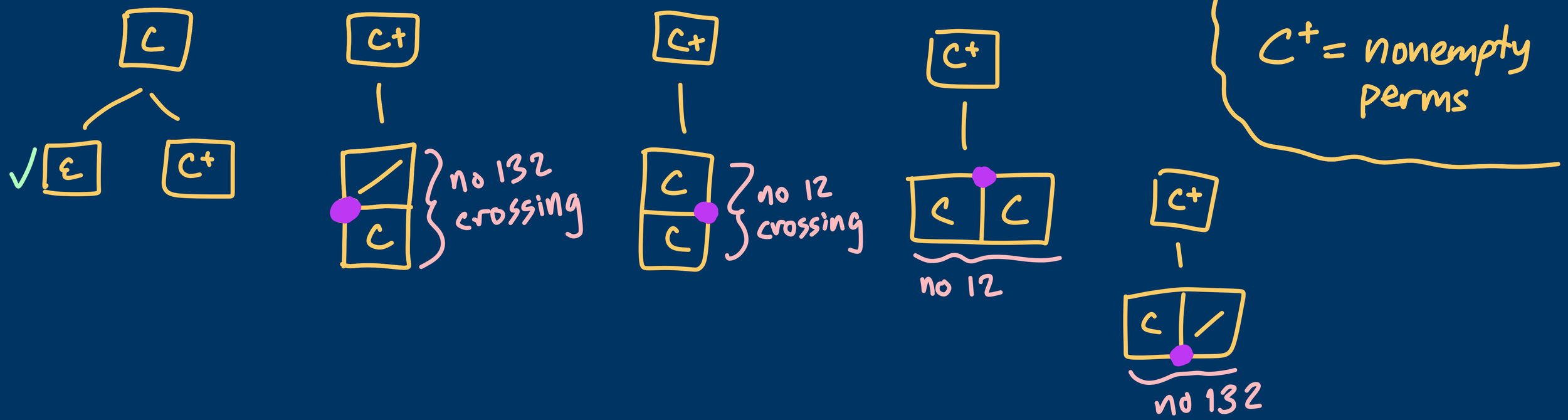

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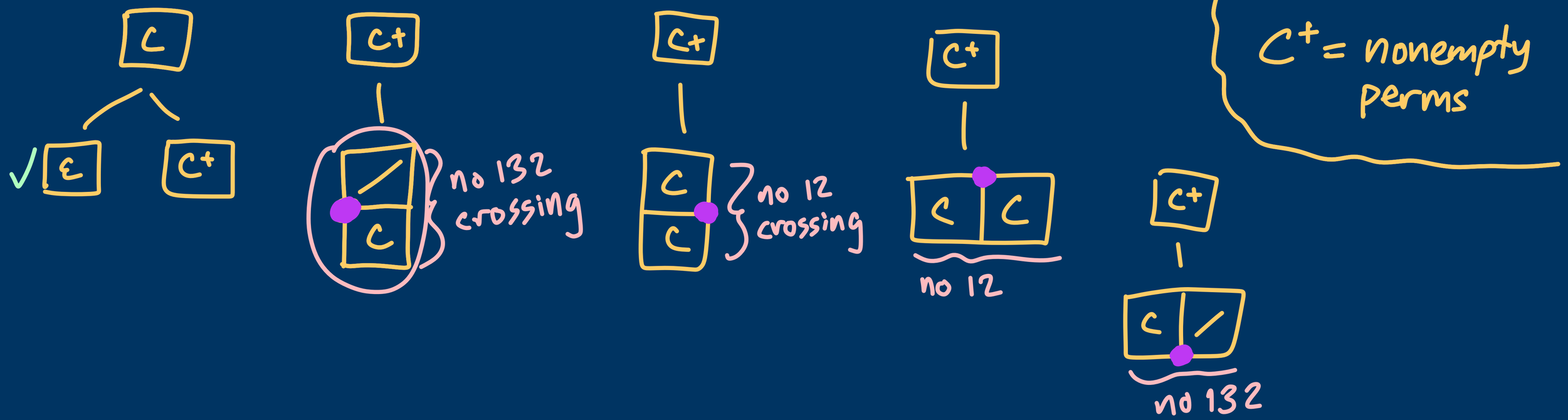
row/col separation

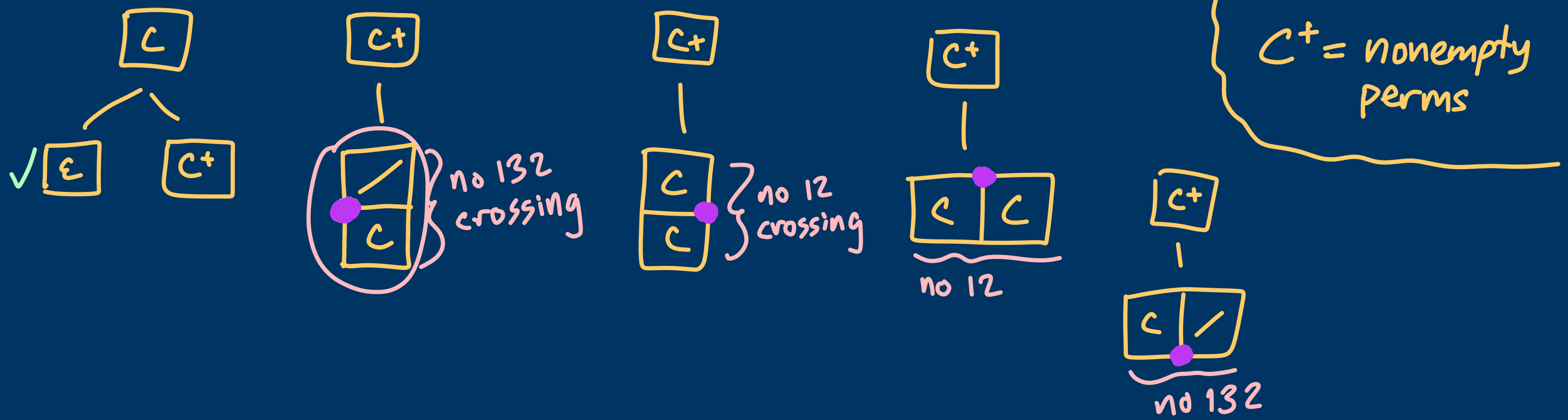
factor



empty or not

point placement

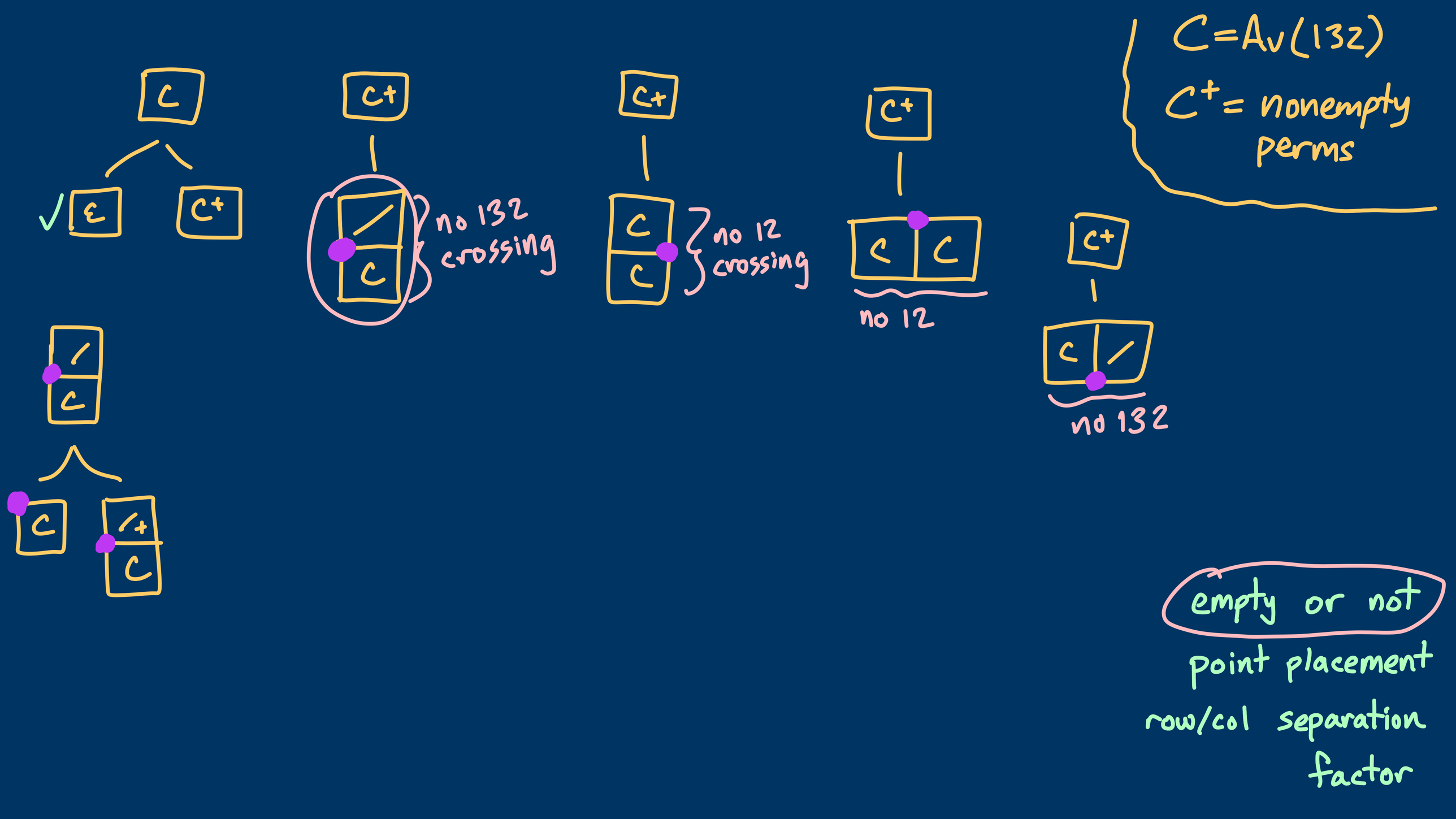
row/col separation factor

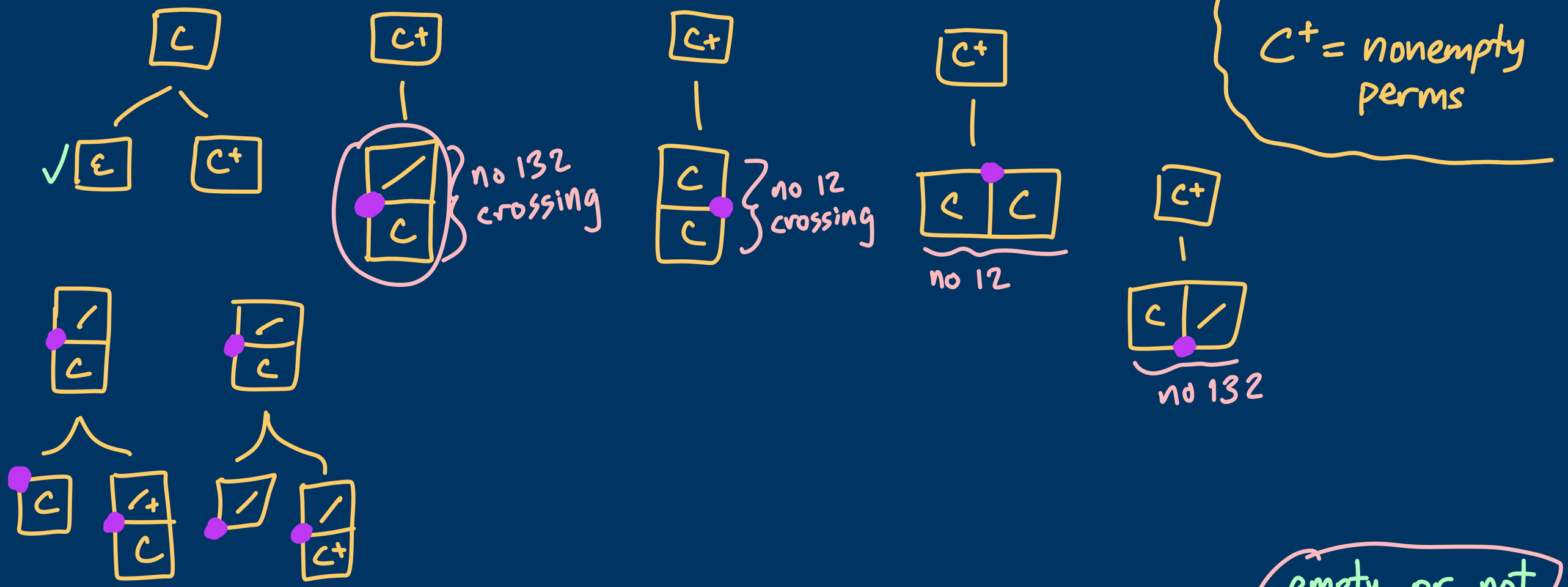


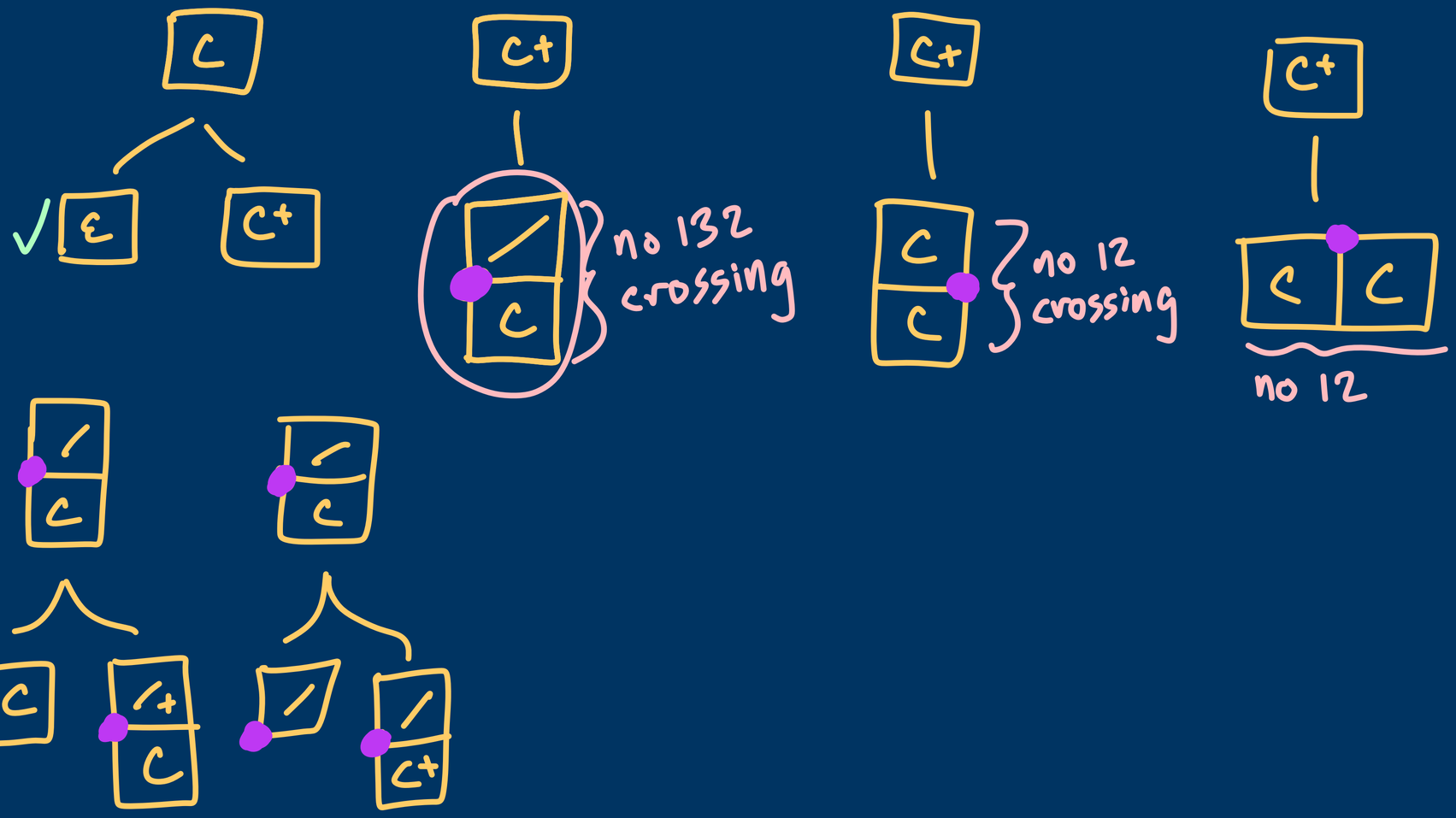
empty or not

point placement

row/col separation factor


C
√ε
C+
C+
no 132 crossing
C+
no 12 crossing
C+
no 12
C+
no 132
C=Av(132)
C+= nonempty perms
empty or not
point placement
row/col separation
factor


C
√ ε
C+
C+
C
C
no 132 crossing
C+
C
C
no 12 crossing
C+
C C
no 12
C+
C
no 132
C=Av(132)
C+ = nonempty perms
empty or not
point placement
row/col separation
factor



$C = Av(132)$

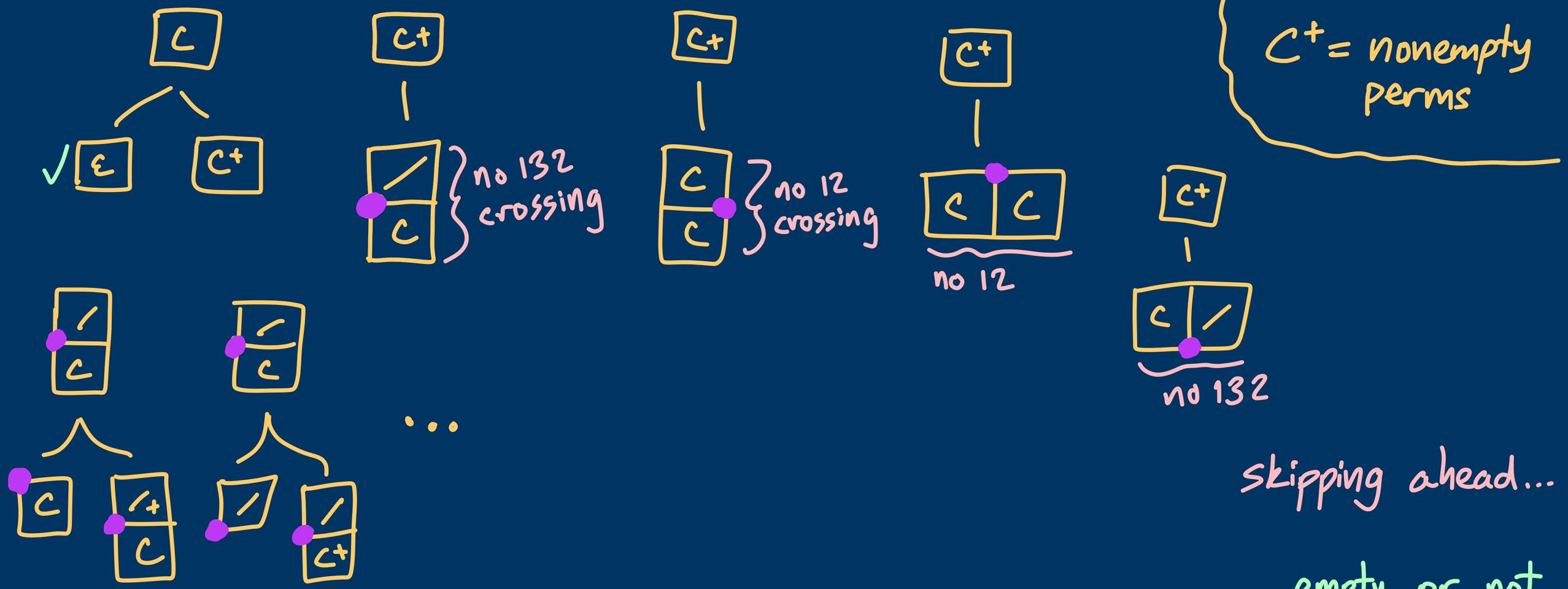
C^+ = nonempty perms

skipping ahead...

empty or not

point placement

row/col separation factor



$C = Av(132)$

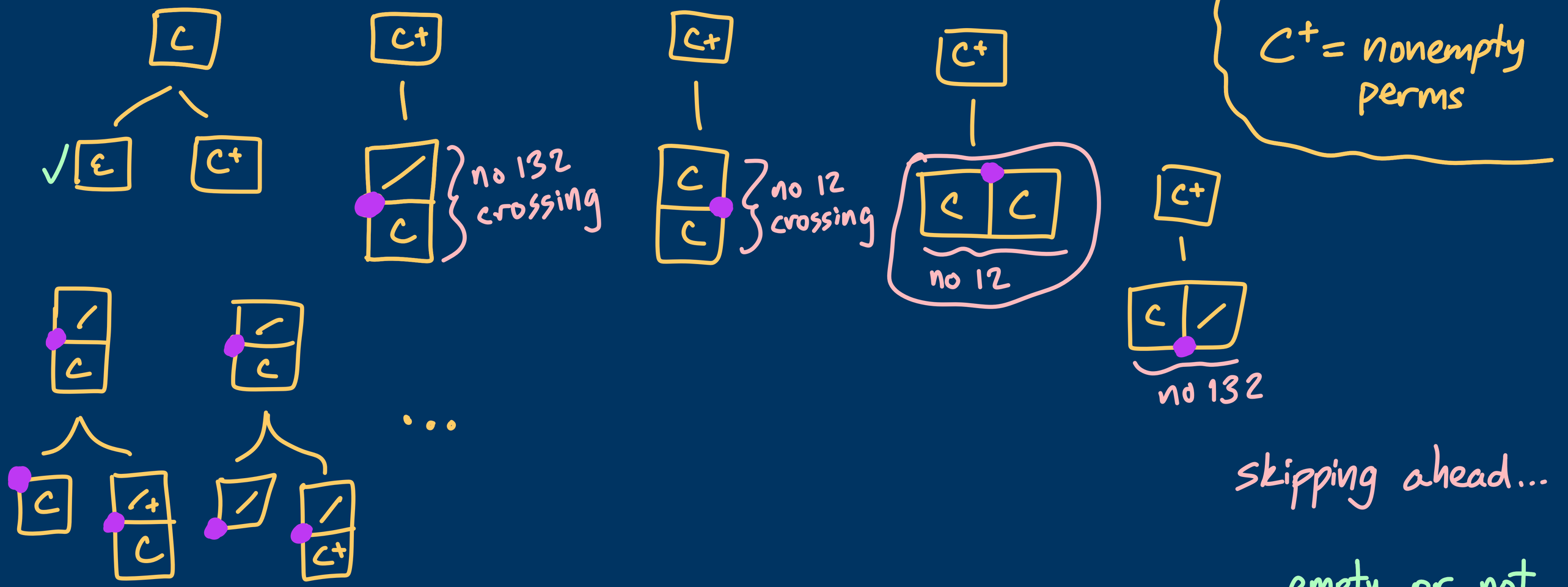
C^+ = nonempty perms

skipping ahead...

empty or not

point placement

row/col separation factor



$C = Av(132)$

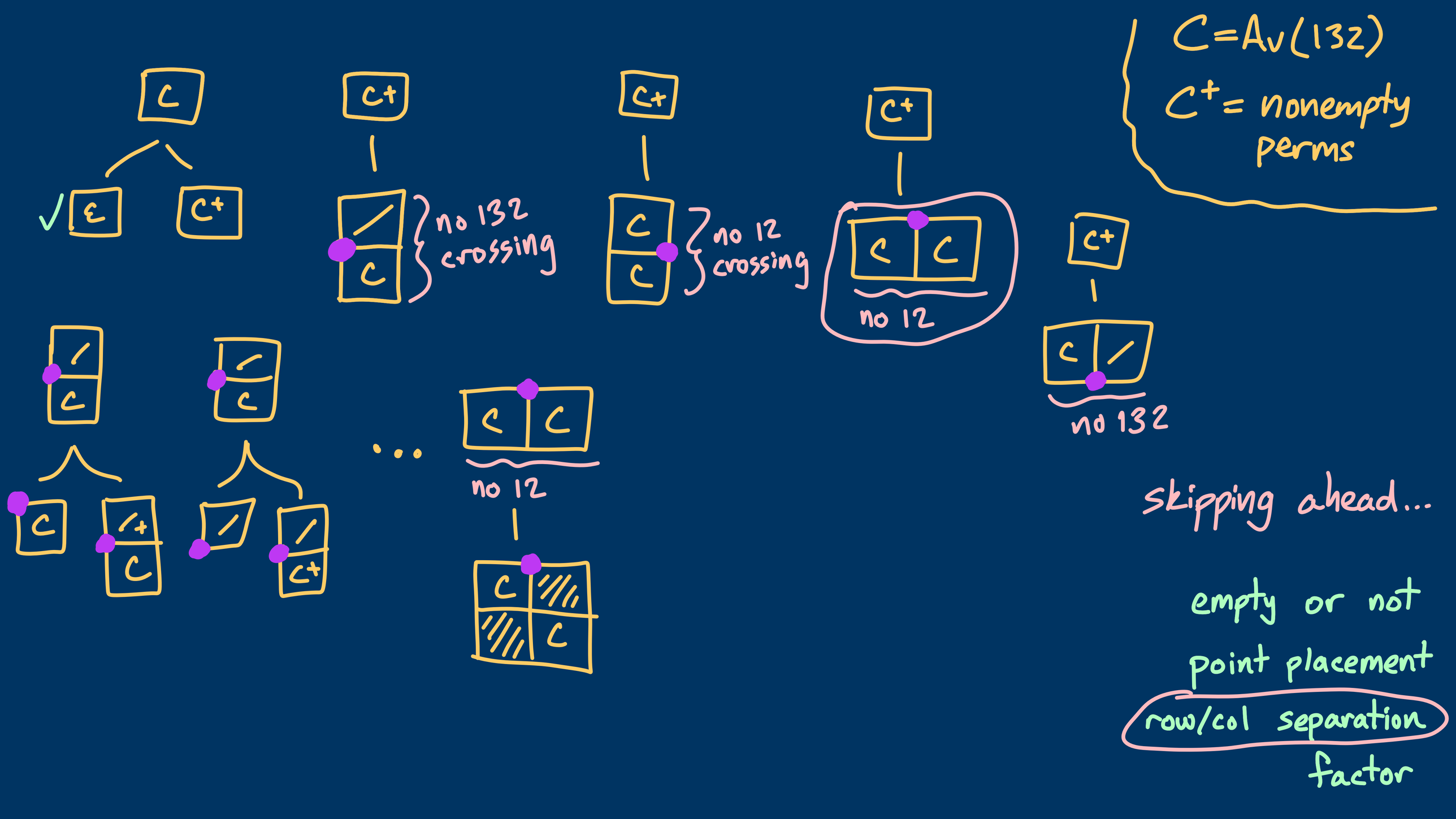
C^+ = nonempty perms

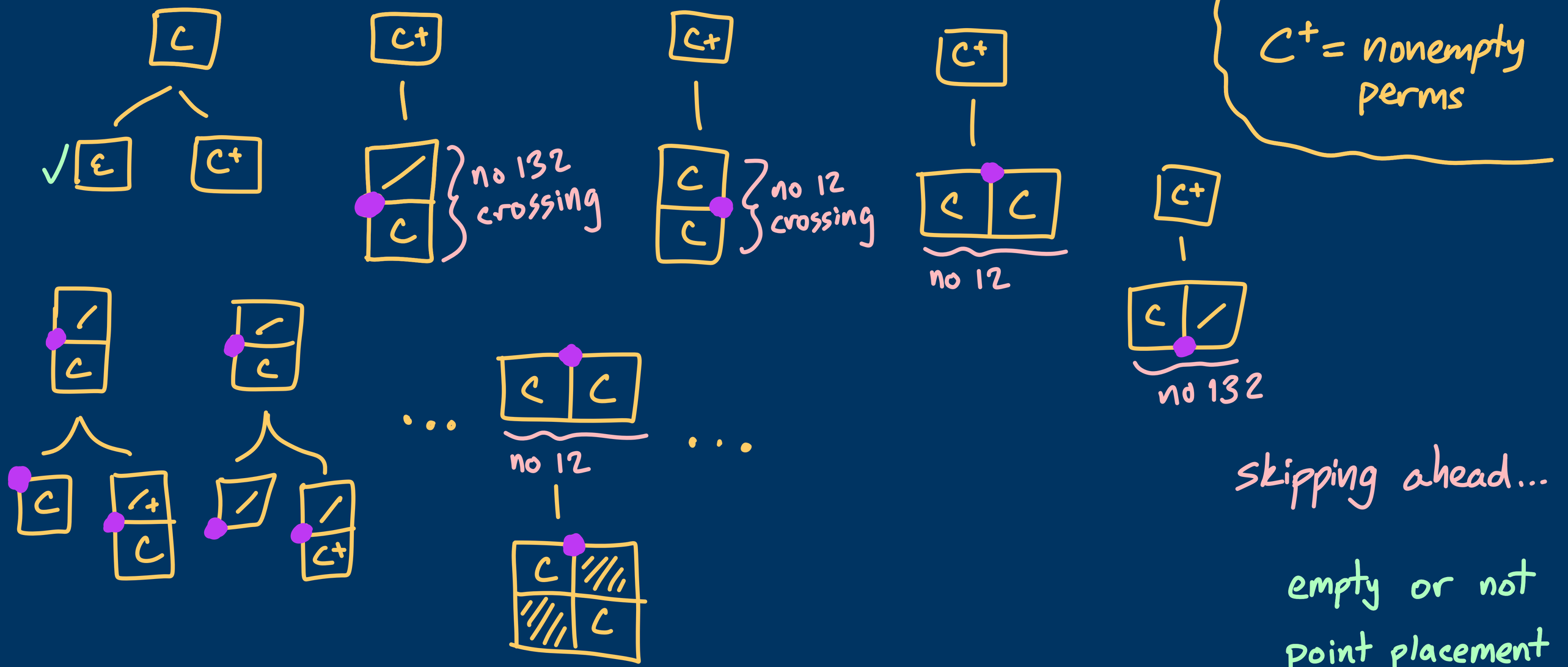
skipping ahead...

empty or not

point placement

row/col separation factor


C
C+
ε
no 132 crossing
no 12 crossing
no 12
no 132
...
C=Av(132)
C+= nonempty perms
skipping ahead...
empty or not
point placement
row/col separation
factor


C
C+
ε
no 132 crossing
no 12 crossing
no 12
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C=Av(132)
C+= nonempty perms
skipping ahead...
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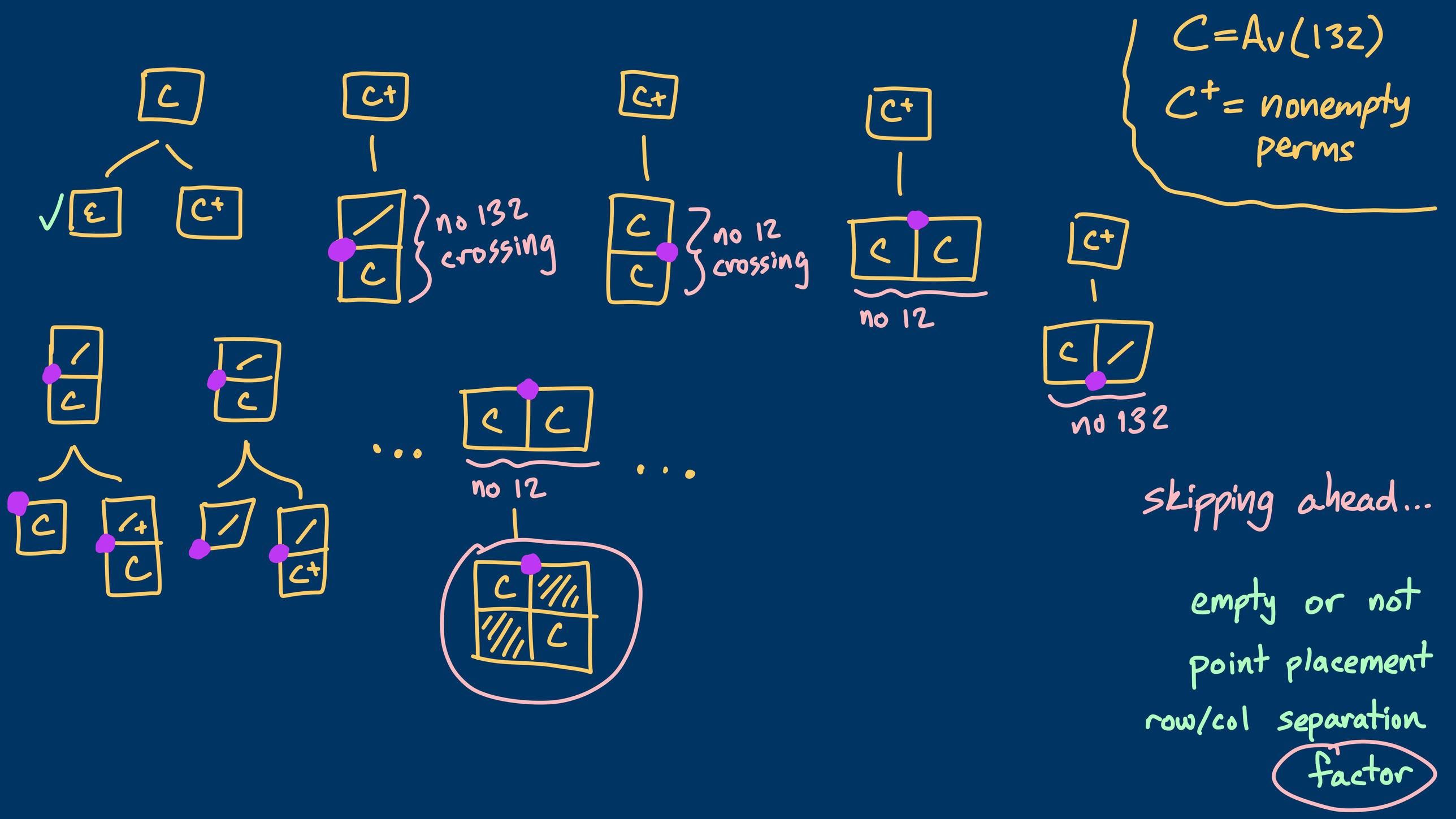
factor


C
C+
ε
no 132 crossing
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C=Av(132)
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skipping ahead...
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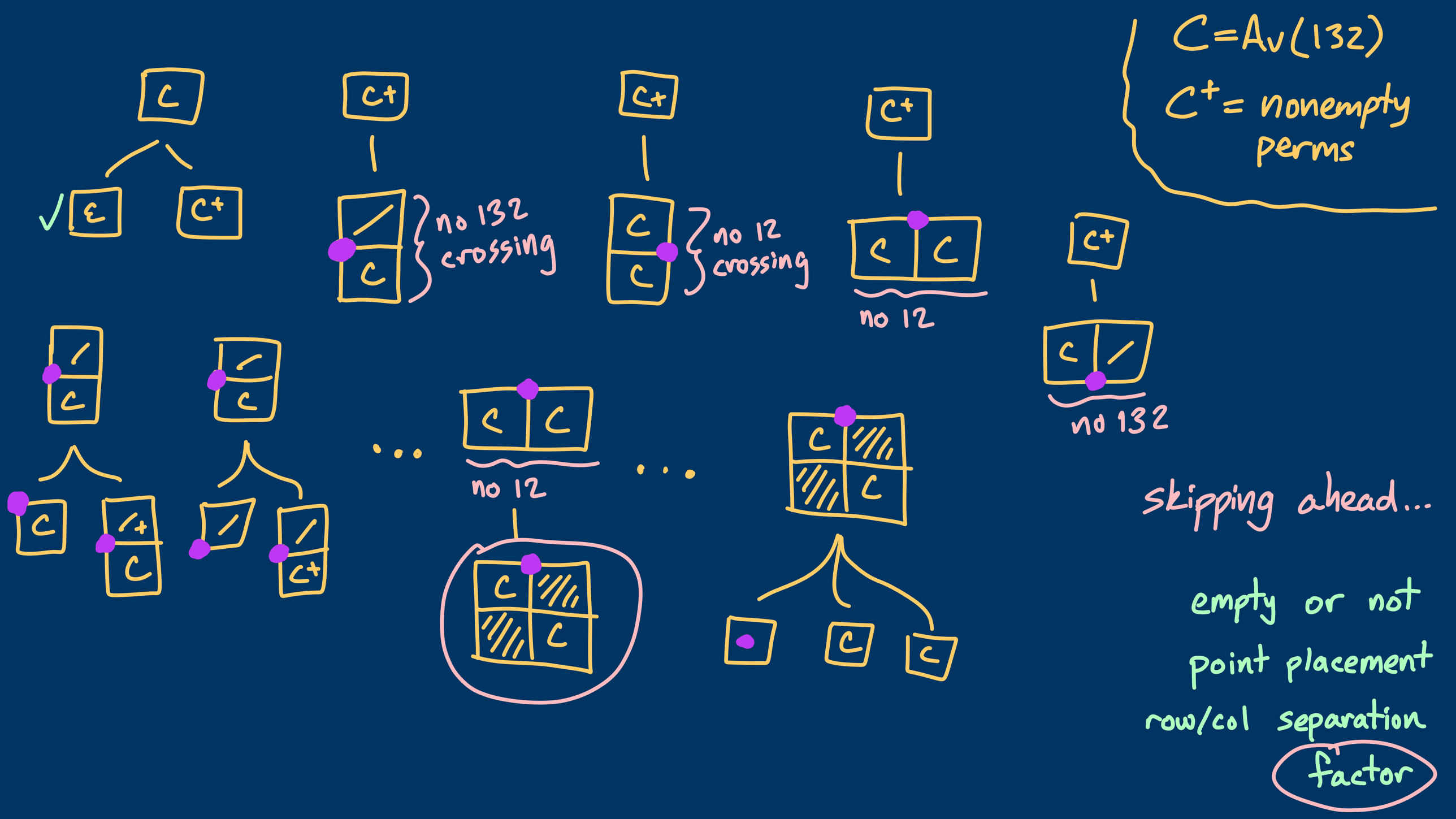
C
C+
ε
no 132 crossing
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C=Av(132)
C+= nonempty perms
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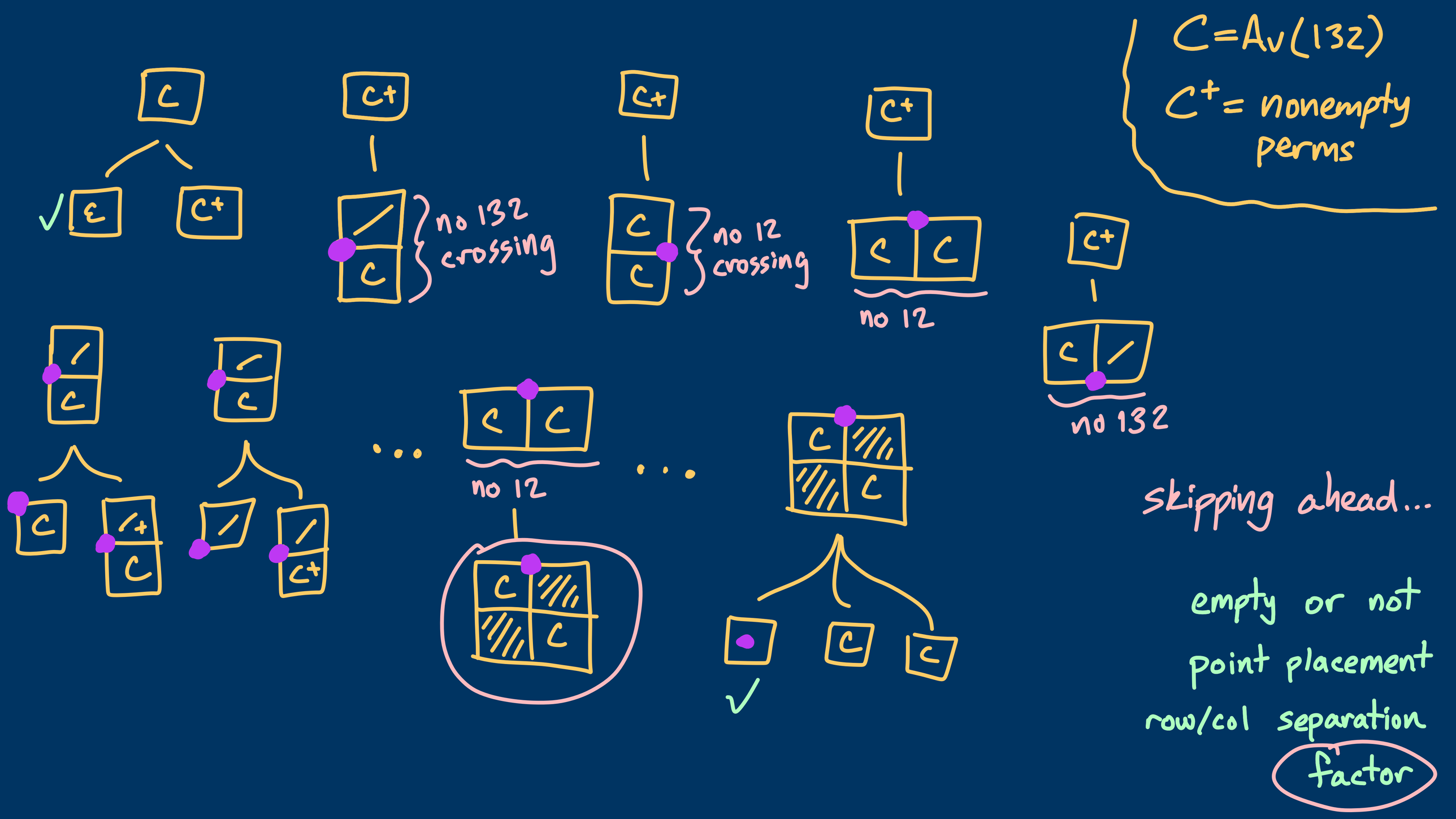
C
C+
ε
no 132 crossing
no 12 crossing
no 12
no 132
C=Av(132)
C+= nonempty perms
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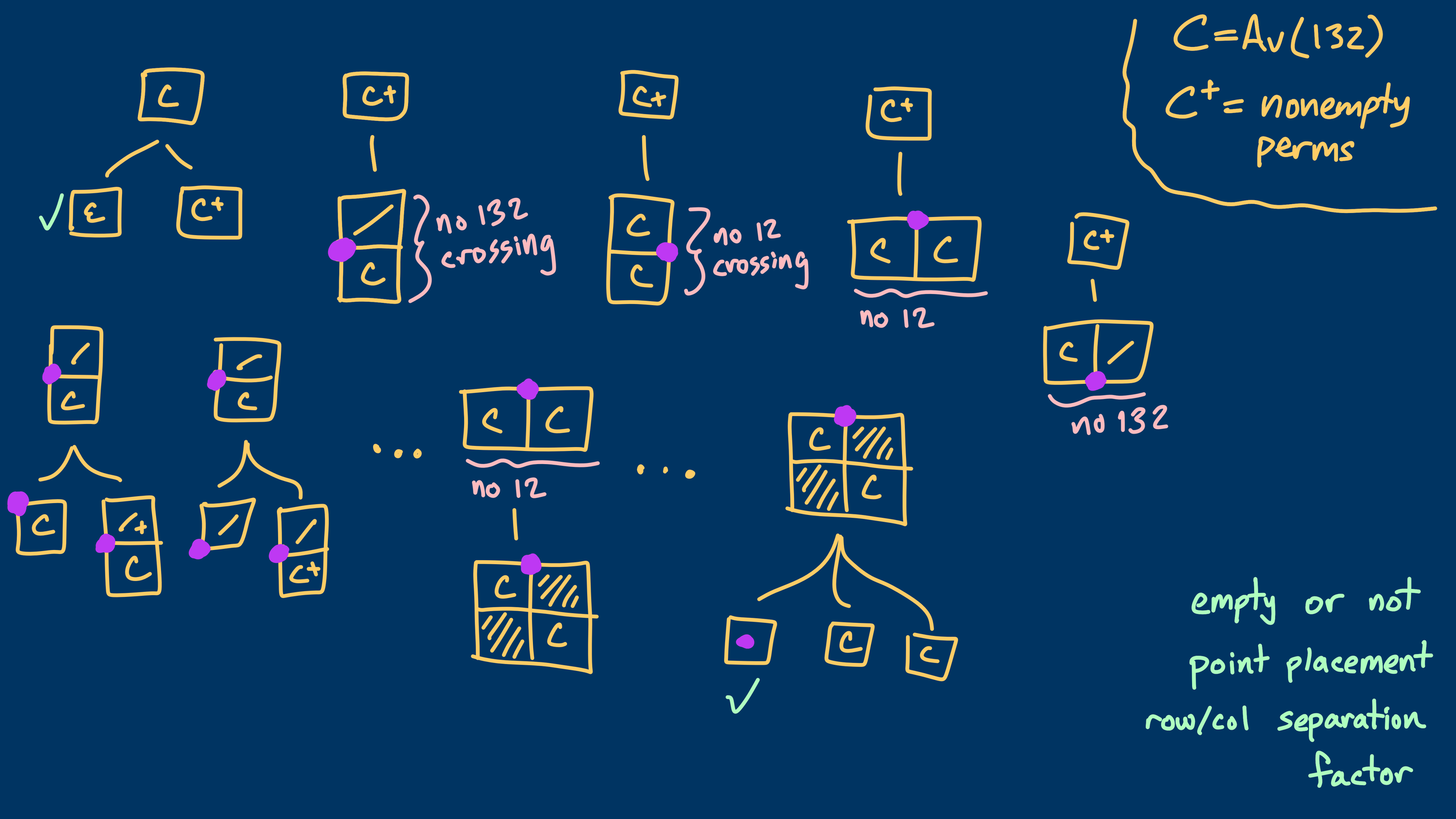
factor


C=Av(132)
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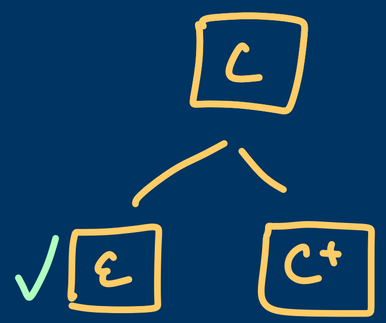
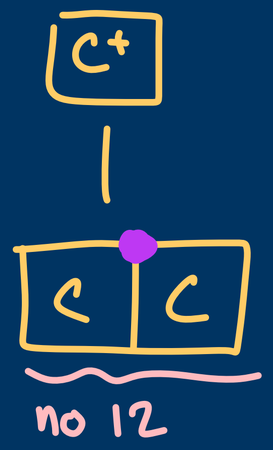
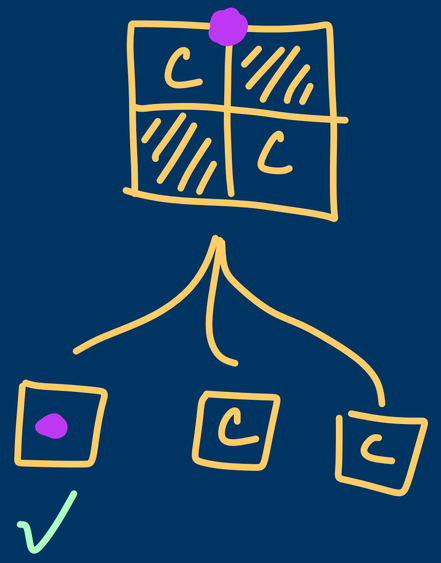
C
C+
ε
no 132 crossing
no 12 crossing
no 12
no 132
empty or not
point placement
row/col separation
factor





$C = Av(132)$

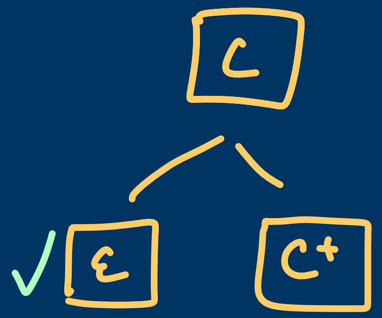
C^+ = nonempty perms

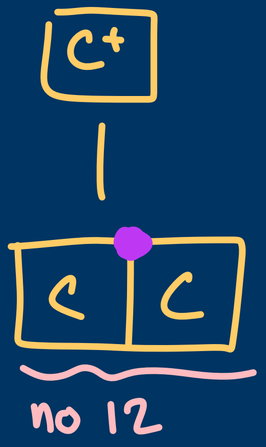


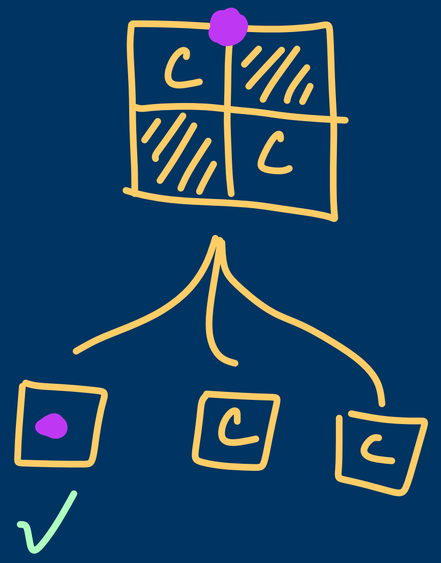
empty or not

point placement

row/col separation factor







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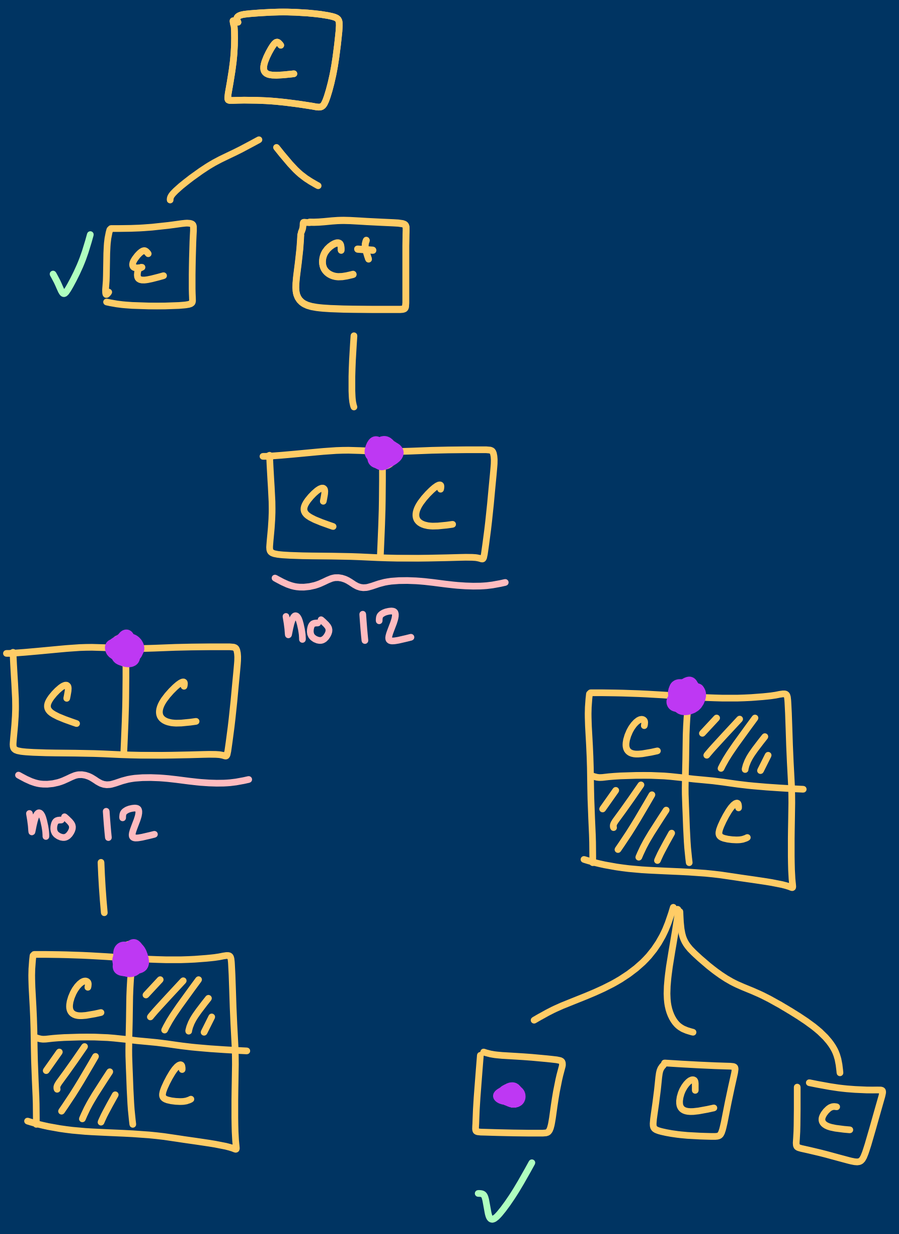
empty or not

point placement

row/col separation factor



$C = Av(132)$

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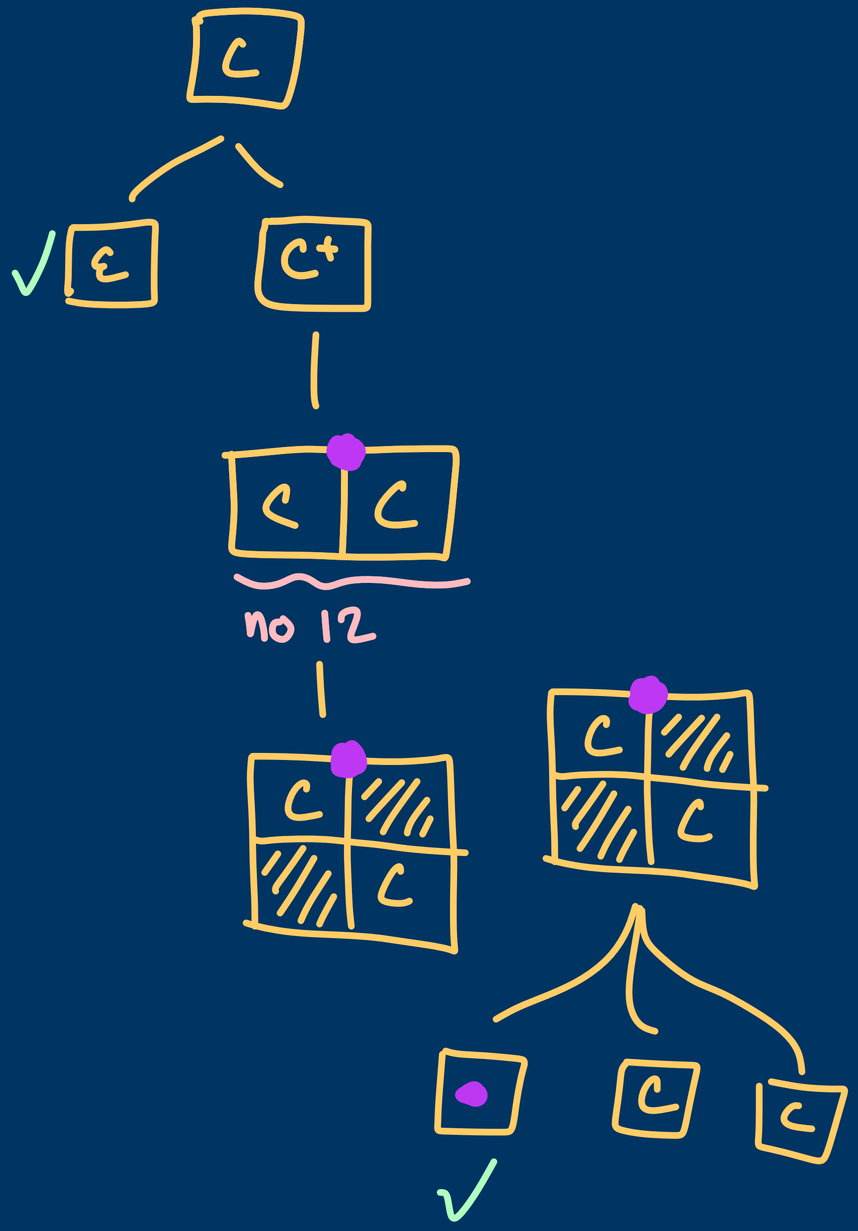
empty or not

point placement

row/col separation factor



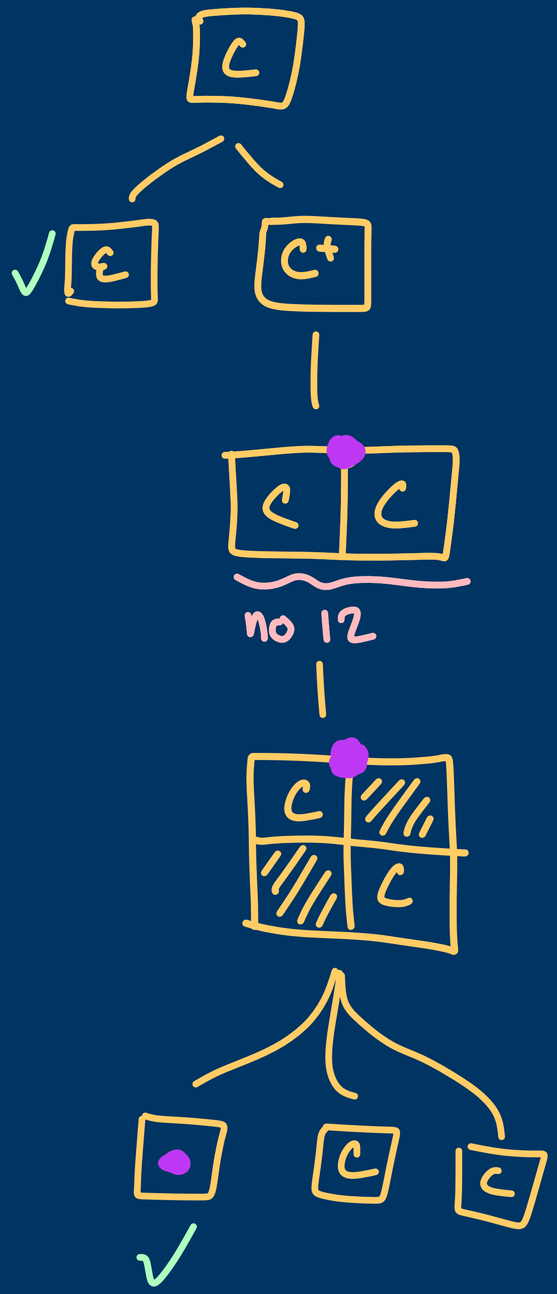
$C = Av(132)$

C^+ = nonempty perms

empty or not

point placement

row/col separation factor



$C = Av(132)$

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empty or not

point placement

row/col separation factor

Combinatorial Exploration

General outline:

- Teach the computer a set of strategies.

Combinatorial Exploration

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- Apply them to the set of permutations you want to enumerate.

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  and then to the children they produced

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  …

Combinatorial Exploration

General outline:

- Teach the computer a set of strategies.
- Apply them to the set of permutations you want to enumerate.
  and then to the children they produced
  and then to the children they produced
  and then to the children they produced
  …
- Each time you apply a strategy to a set, you make a puzzle piece. Search the pile of puzzle pieces for a subset that makes a combinatorial specification. If you find one, you win!
  polynomial-time counting algorithm, system of equations for the GF,
  uniform random sampling routine, exhaustive generation (but slow)

Combinatorial Exploration

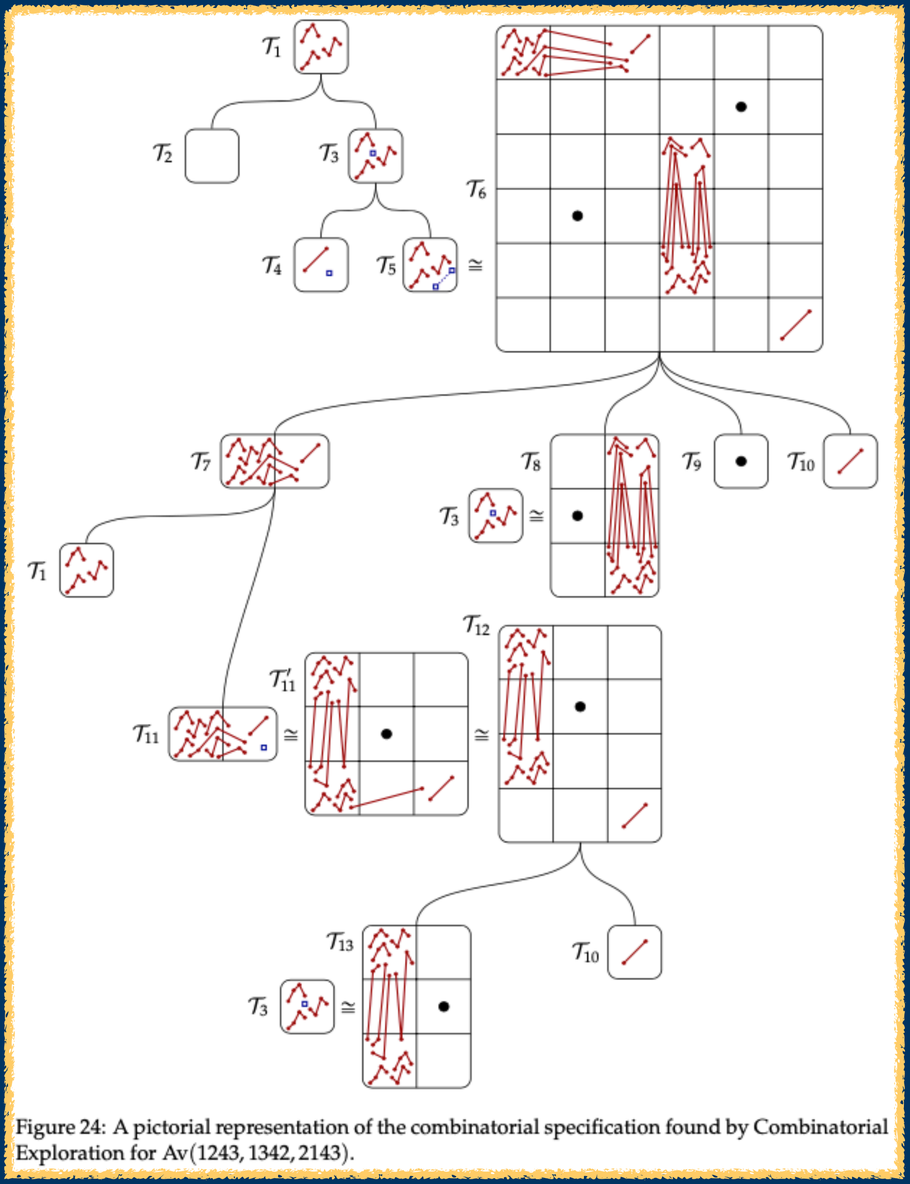


Figure 24: A pictorial representation of the combinatorial specification found by Combinatorial Exploration for Av(1243, 1342, 2143).

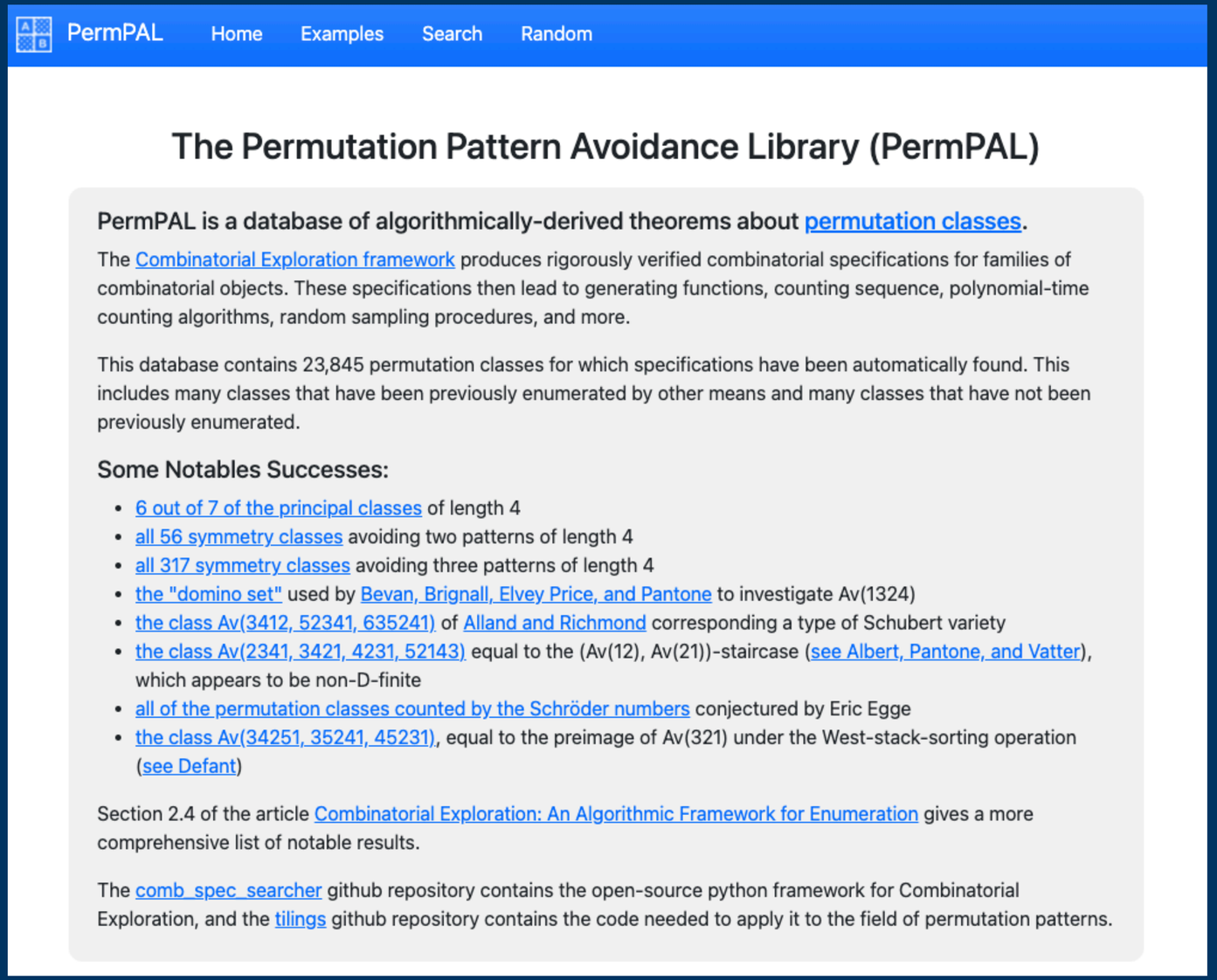
Av(1243, 1342, 2143)

The algorithm generates about 5,400 rules before it finds this subset of 10 rules that makes a rigorous specification.

55 seconds

Combinatorial Exploration

https://permpal.com



Combinatorial Exploration

We can find combinatorial specifications for:

- 6 out of 7 of the classes avoiding 1 pattern of length 4
  First direct enumerations of $\mathrm{Av}(1342)$ and $\mathrm{Av}(2413)$

- All 56 classes avoiding 2 patterns of length 4
  3 are conjectured to be non-D-finite
  can derive the algebraic GF for the other 53

- All 317 classes avoiding 3 patterns of length 4

- All classes avoiding 4 or more patterns of length 4

Combinatorial Exploration

We can find combinatorial specifications for:

- 1324-avoiding domino permutations
- Preimage of $\mathrm{Av}(321)$ under West-stack-sorting
  $\mathrm{Av}(34251, 35241, 45231)$
- LCI Schubert Varieties
  $\mathrm{Av}(52341, 53241, 52431, 35142, 42513, 351624)$
- “Box classes” like $\mathrm{Av}(1\square 2\square 3)$ and $\mathrm{Av}(1\square\square 32)$
- “POP classes”
- Permutations corresponding to Schubert varieties with a complete parabolic bundle structure
  $\mathrm{Av}(3412, 52341, 635241)$

Av(2143, 3412)

View Raw Data

Generating Function

$$\frac{3x-1}{\sqrt{-4x+1}\,(2x-1)}$$

Copy to clipboard: latex Maple sympy Search on PermPAL

Counting Sequence

1, 1, 2, 6, 22, 86, 340, 1340, 5254, 20518, 79932, 311028, 1209916, 4707964, 18330728, ...

Copy 101 terms to clipboard Search on OEIS Search on PermPAL

Recurrence

$$a(0) = 1$$
$$a(1) = 1$$
$$a(2) = 2$$
$$a(n+3) = \frac{12\,(1+2n)a(n)}{n+3} - \frac{2\,(16+13n)a(n+1)}{n+3} + \frac{(19+9n)a(n+2)}{n+3},$$

Copy to clipboard: latex Maple

Implicit Equation for the Generating Function

$$(4x-1)(2x-1)^2F(x)^2 + (3x-1)^2 = 0$$

Copy to clipboard: latex Maple Search on PermPAL

Heatmap

To create this heatmap, we sampled 1,000,000 permutations of length 300 uniformly at random. The color of the point (i, j) represents how many permutations have value j at index i (darker = more).

$$a(n+3) = \frac{12(1+2n)a(n)}{n+3} - \frac{2(16+13n)a(n+1)}{n+3} + \frac{(19+9n)a(n+2)}{n+3},$$

Copy to clipboard: latex Maple

Heatmap

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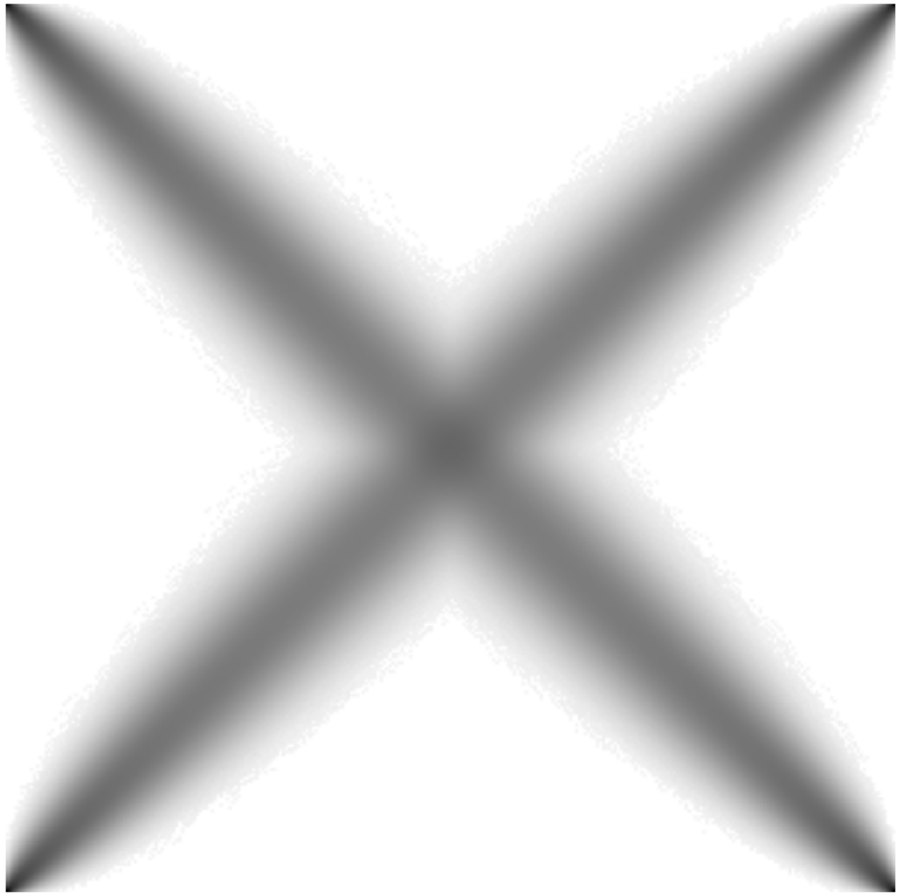


Specification 1 | Specification 2 | Specification 3 | Specification 4 | Specification 5

This specification was found using the strategy pack "Row And Col Placements Tracked Fusion Isolated" and has 29 rules.

Found on April 21, 2021.

Specification 1 | Specification 2 | Specification 3 | Specification 4 | Specification 5

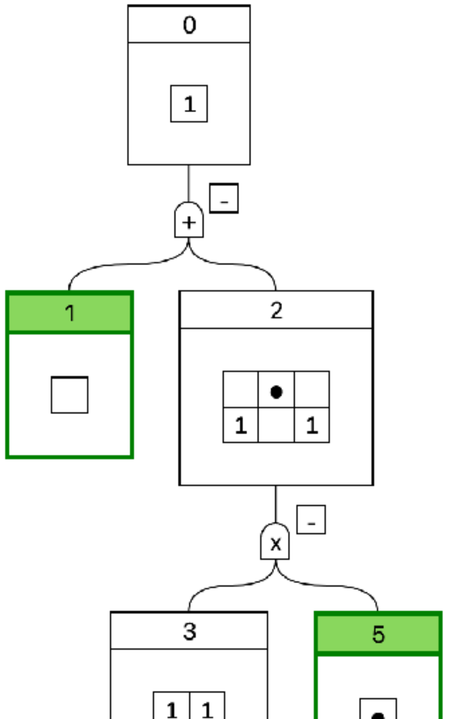
This specification was found using the strategy pack "Row And Col Placements Tracked Fusion Isolated" and has 29 rules.

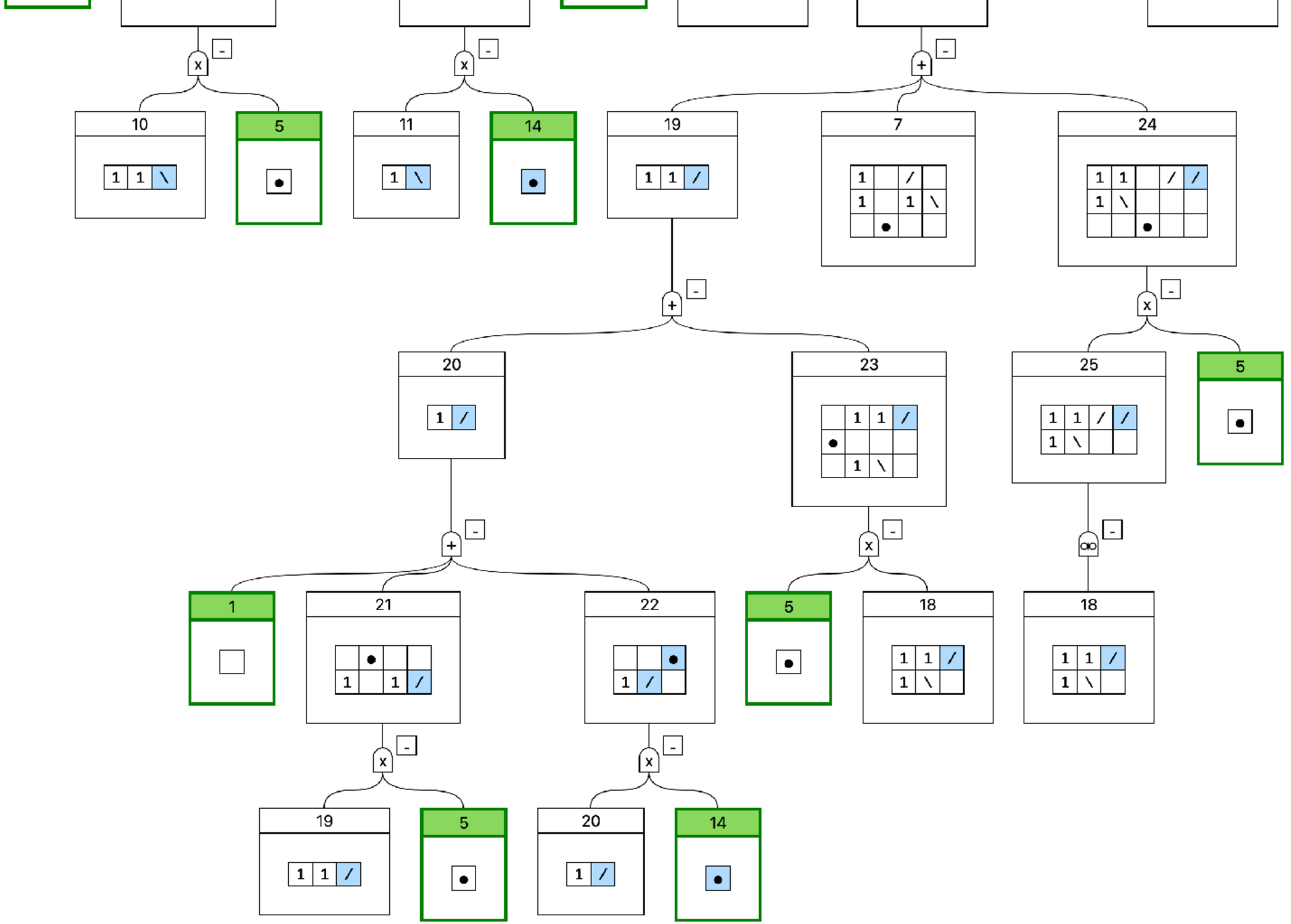
Found on April 21, 2021.
Finding the specification took 653 seconds.

Proof Tree

Copy to clipboard: specification json | pack json

View tree on standalone page.

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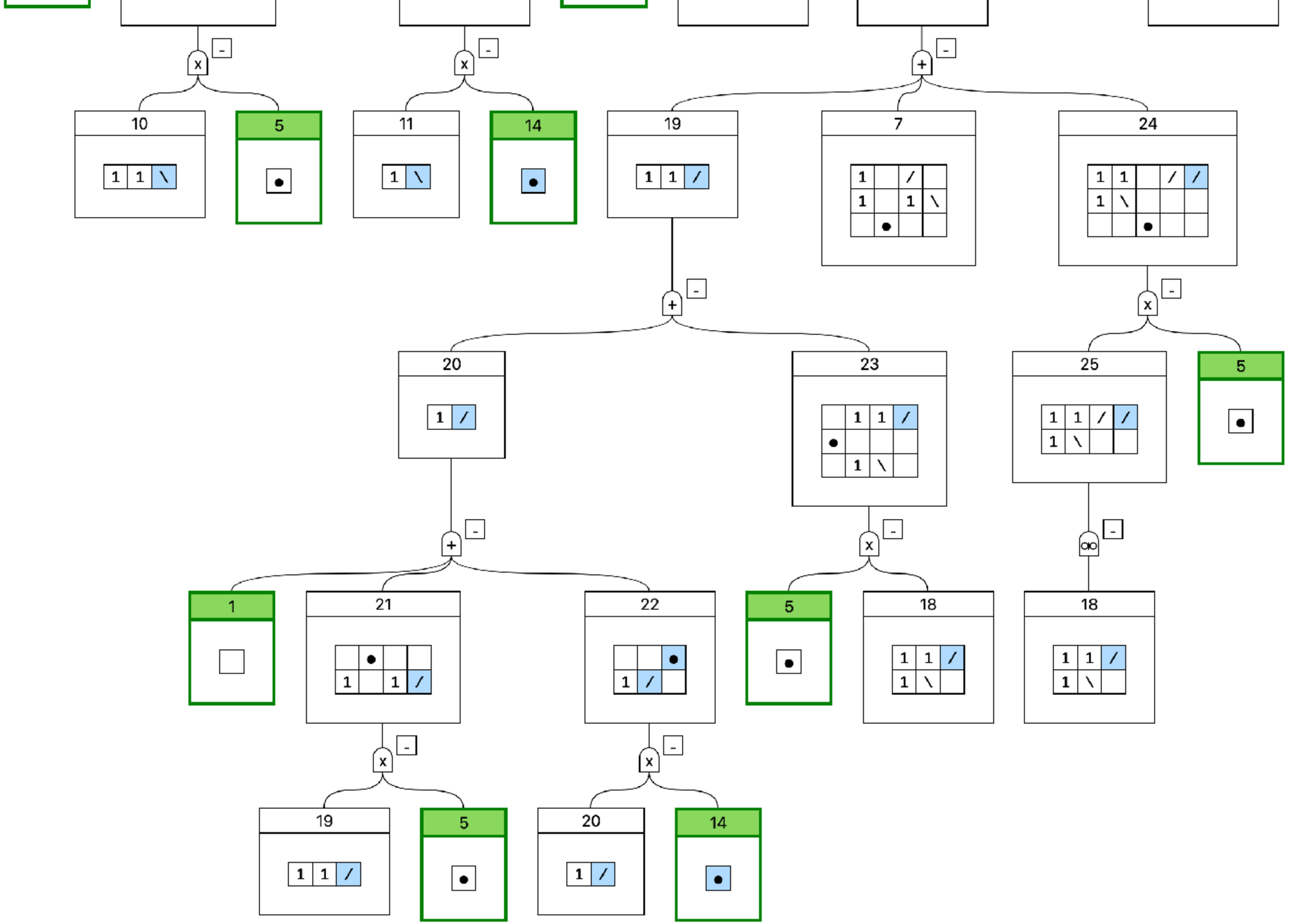
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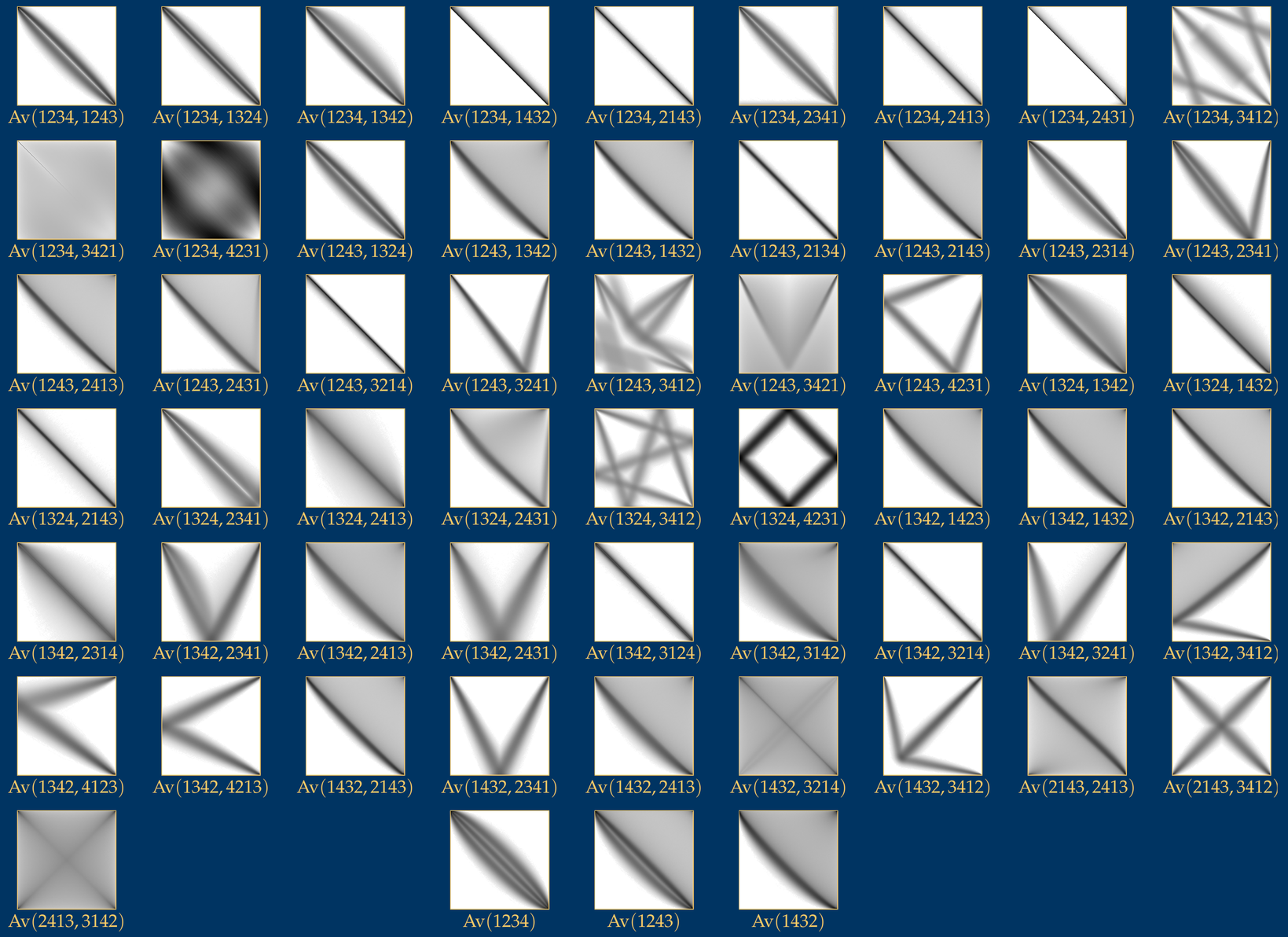
14

System of Equations

Copy 29 equations to clipboard: latex | Maple | sympy

$$
\begin{aligned}
F_0(x) &= F_1(x) + F_2(x) \\
F_1(x) &= 1 \\
F_2(x) &= F_3(x)F_5(x) \\
F_3(x) &= F_0(x) + F_4(x) \\
F_4(x) &= F_5(x)F_6(x) \\
F_5(x) &= x \\
F_6(x) &= F_{28}(x) + F_3(x) + F_7(x) \\
F_7(x) &= F_5(x)F_8(x) \\
F_8(x) &= F_9(x, 1) \\
F_9(x, y) &= F_{10}(x, y) + F_{16}(x) + F_{26}(x, y) \\
F_{10}(x, y) &= F_{11}(x, y) + F_{15}(x, y) \\
F_{11}(x, y) &= F_1(x) + F_{12}(x, y) + F_{13}(x, y) \\
F_{12}(x, y) &= F_{10}(x, y)F_5(x) \\
F_{13}(x, y) &= F_{11}(x, y)F_{14}(x, y) \\
F_{14}(x, y) &= yx \\
F_{15}(x, y) &= F_5(x)F_9(x, y) \\
F_{16}(x) &= F_{17}(x)F_5(x) \\
F_{17}(x) &= F_{18}(x, 1) \\
F_{18}(x, y) &= F_{19}(x, y) + F_{24}(x, y) + F_7(x) \\
F_{19}(x, y) &= F_{20}(x, y) + F_{23}(x, y) \\
F_{20}(x, y) &= F_1(x) + F_{21}(x, y) + F_{22}(x, y) \\
F_{21}(x, y) &= F_{19}(x, y)F_5(x) \\
F_{22}(x, y) &= F_{14}(x, y)F_{20}(x, y) \\
F_{23}(x, y) &= F_{18}(x, y)F_5(x) \\
F_{24}(x, y) &= F_{25}(x, y)F_5(x) \\
F_{25}(x, y) &= -\frac{-yF_{18}(x, y) + F_{18}(x, 1)}{-1 + y} \\
F_{26}(x, y) &= F_{27}(x, y)F_5(x) \\
F_{27}(x, y) &= -\frac{-yF_9(x, y) + F_9(x, 1)}{-1 + y} \\
F_{28}(x) &= F_{17}(x)F_5(x)
\end{aligned}
$$

Combinatorial Exploration



Combinatorial Exploration

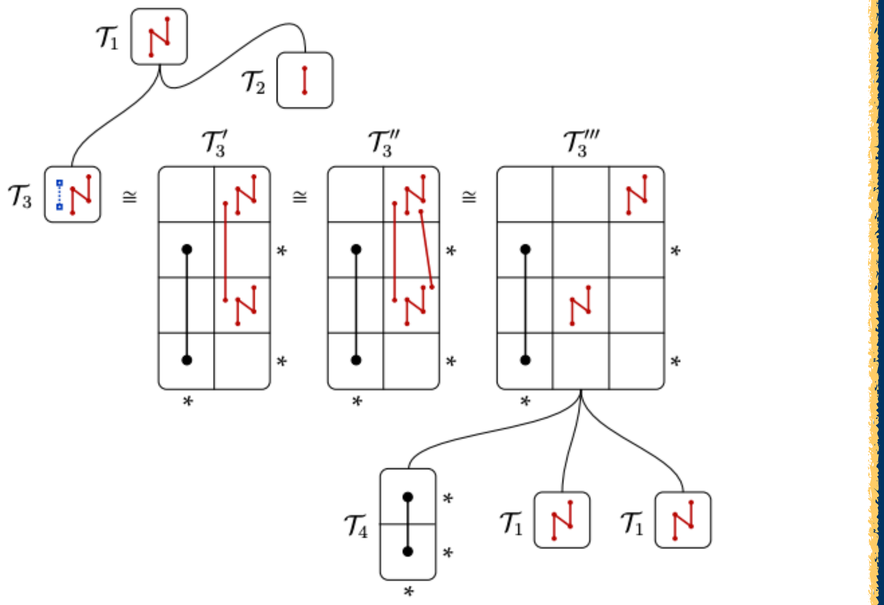


Figure 2.4: A visual representation of the proof tree for noncrossing diagrams $\mathcal{D}(\frown\frown)$.

Enumerative perspectives on chord diagrams

by

Lukas Nabergall

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Combinatorics and Optimization

Combinatorial Exploration

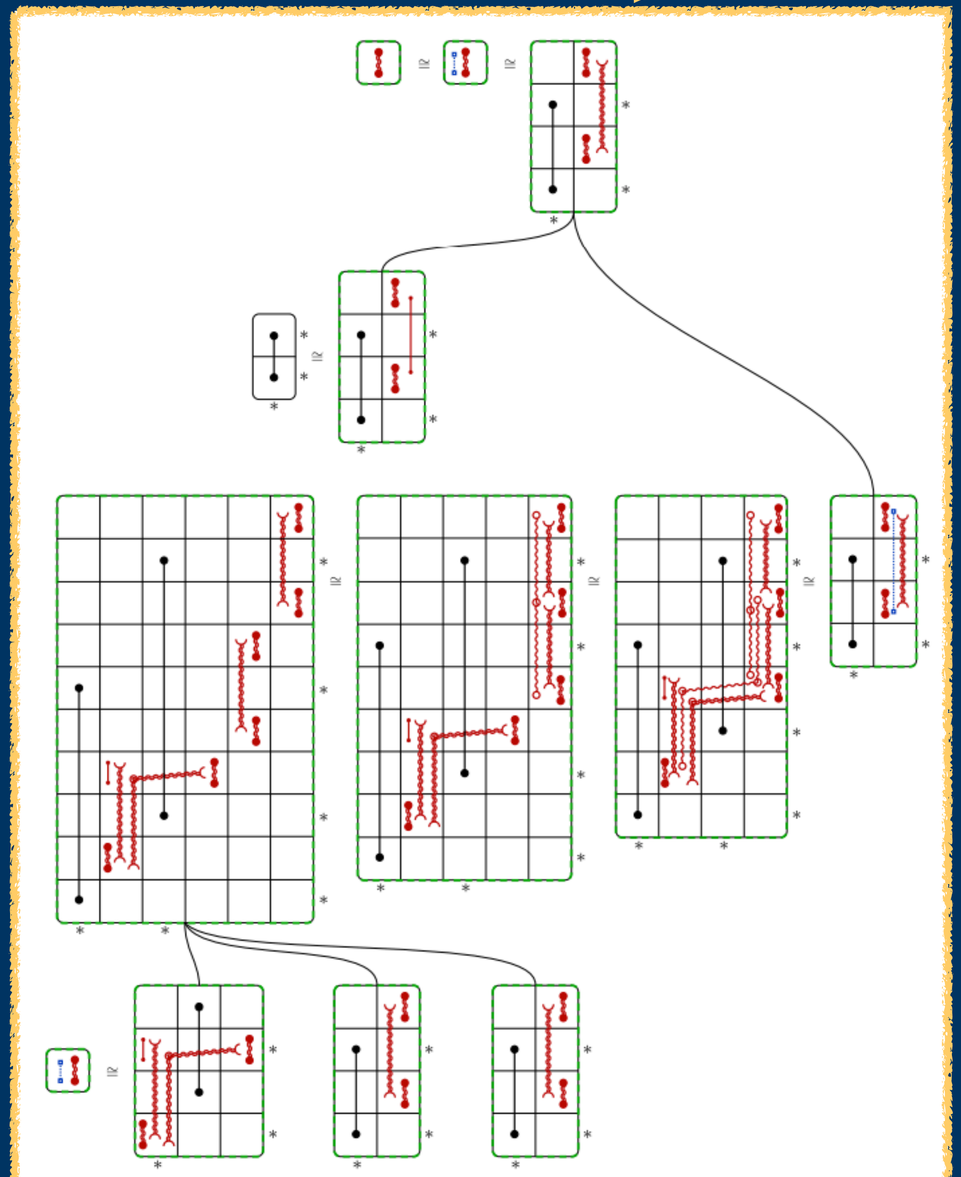


Figure 3.2: A proof tree for the class $\mathcal{C}(T_{\geqslant 3}, B_{\geqslant 3})$ of tree diagrams.

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A thesis
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thesis requirement for the degree of
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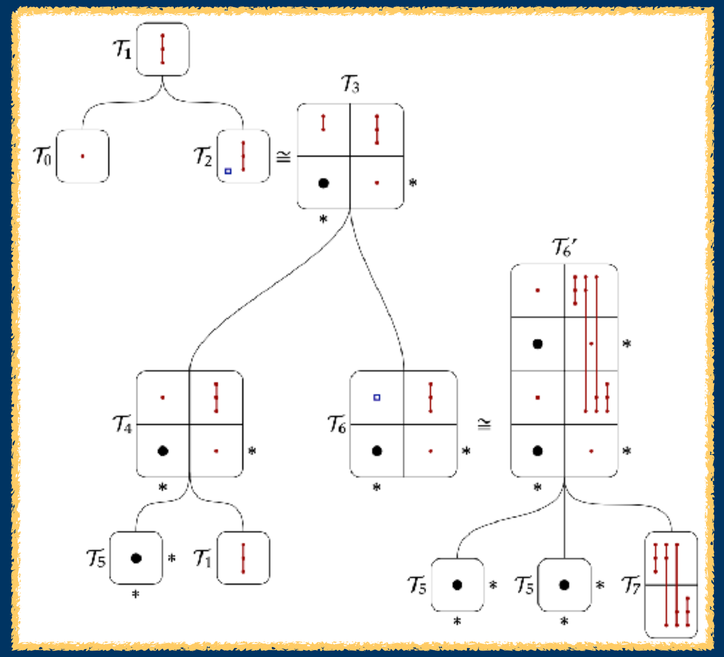
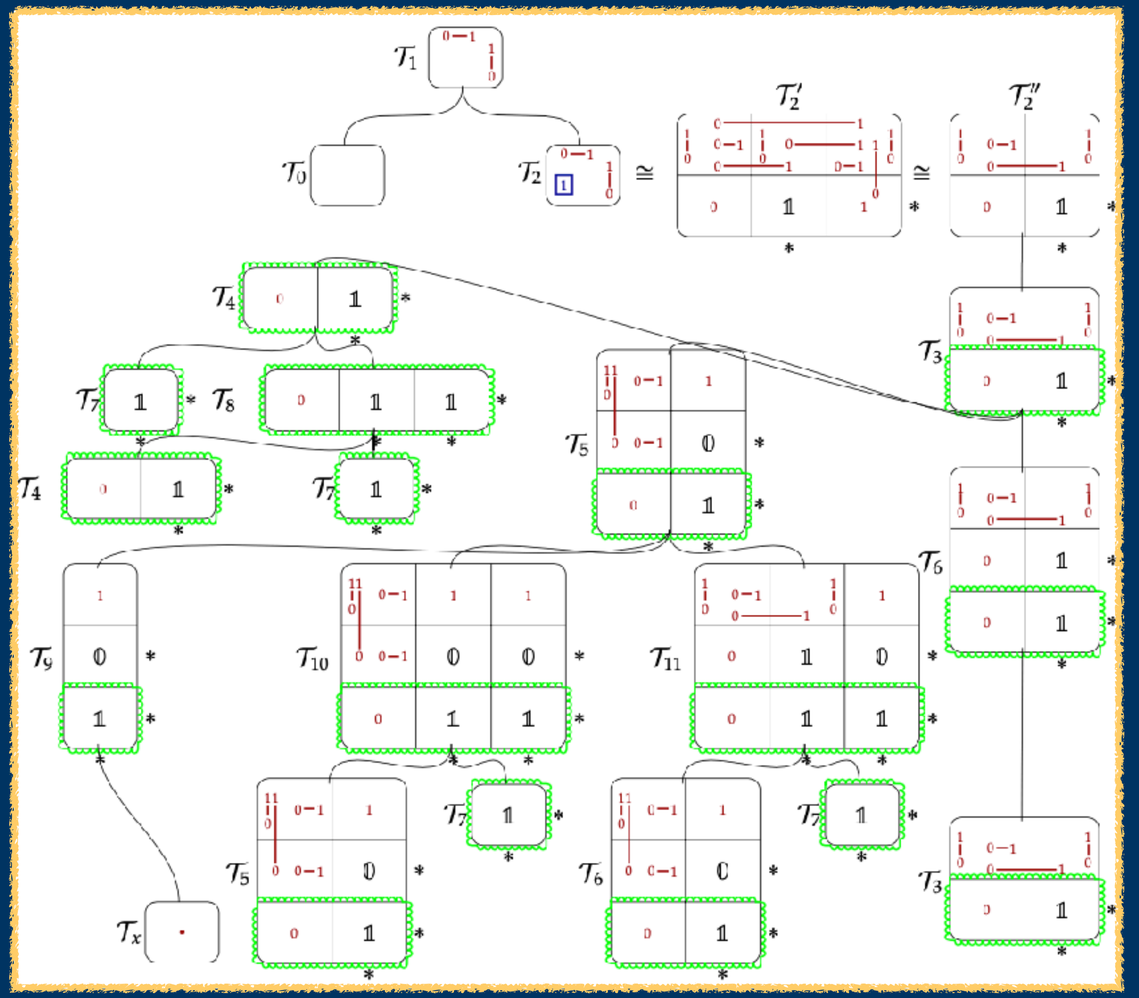
Combinatorial Exploration

Set Partitions



$$T_1(x) = 1 + (x + x^2)T_1(x) + x^3\frac{d}{dx}T_1(x)$$

Polyominoes



$$T_1(x) = \prod_{i=1}^{\infty} \frac{1}{1 - x^i}$$

	rigorous	non-rigorous
experimental	- enumeration schemes WILF, WILFPLUS, Flexible Schemes (E) - Combinatorial Exploration (E)	- Struct - BiSC
non-experimental	- generating trees (E) - FINLABEL - ECO Method - Combinatorial Generation - Regular Insertion Enc. - Finite Simples - Poly Classes	- HERB

	rigorous	non-rigorous
experimental	- enumeration schemes WILF, WILFPLUS, Flexible Schemes (E) - Combinatorial Exploration (E)	- Struct - BiSC - GuessFunc
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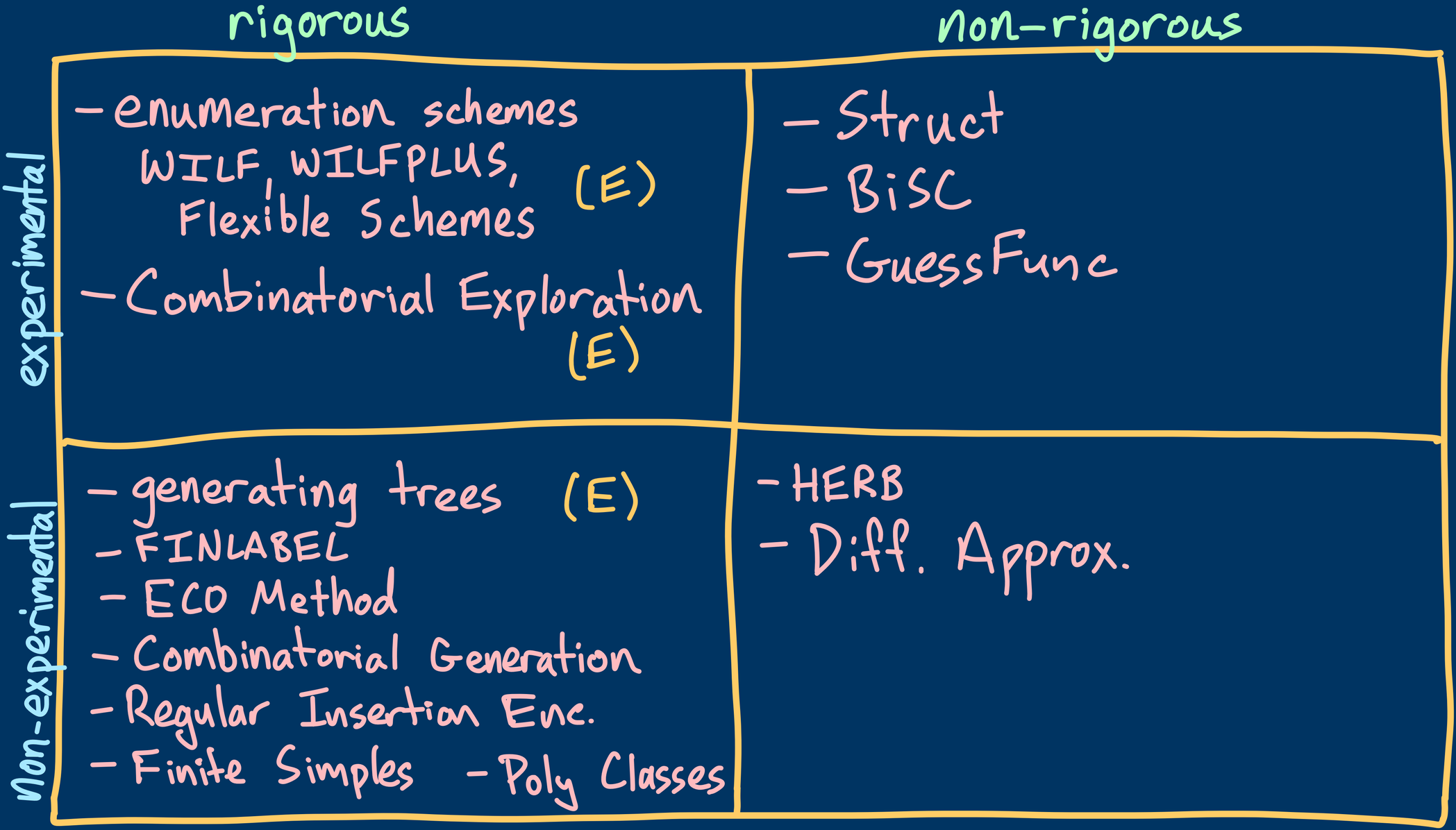
rigorous
non-rigorous
experimental
non-experimental
- enumeration schemes
WILF, WILFPLUS,
Flexible Schemes (E)
- Combinatorial Exploration (E)
- Struct
- BiSC
- GuessFunc
- generating trees (E)
- FINLABEL
- ECO Method
- Combinatorial Generation
- Regular Insertion Enc.
- Finite Simples
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- HERB
- Diff. Approx.

PermLab

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TTDD

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TTDD

Kuszmaul

28 years ago...

Enumeration Schemes and, More Importantly, Their Automatic Generation

Doron Zeilberger*

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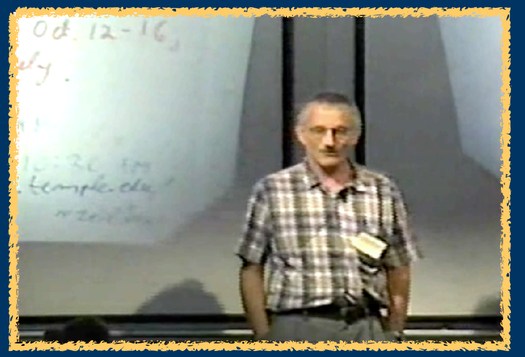
Received May 27, 1998



Apology. The success rate of the present method, in its present state, is somewhat disappointing. Ekhad was able to reproduce the classical cases and a few new ones, but for most patterns and sets of patterns, it failed to find a scheme (defined below) of reasonable depth. But the present framework for setting up a scheme could be modified and extended in various ways. We do believe that an appropriate enhancement of the present method would yield, if not a 100% success rate, at least close to it.

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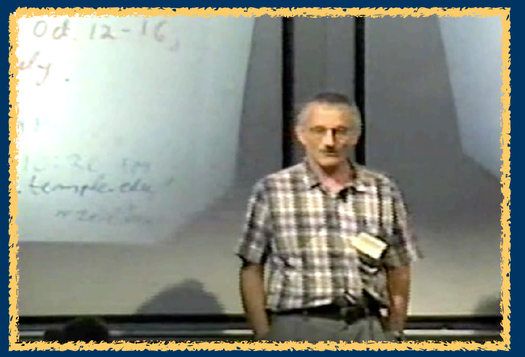

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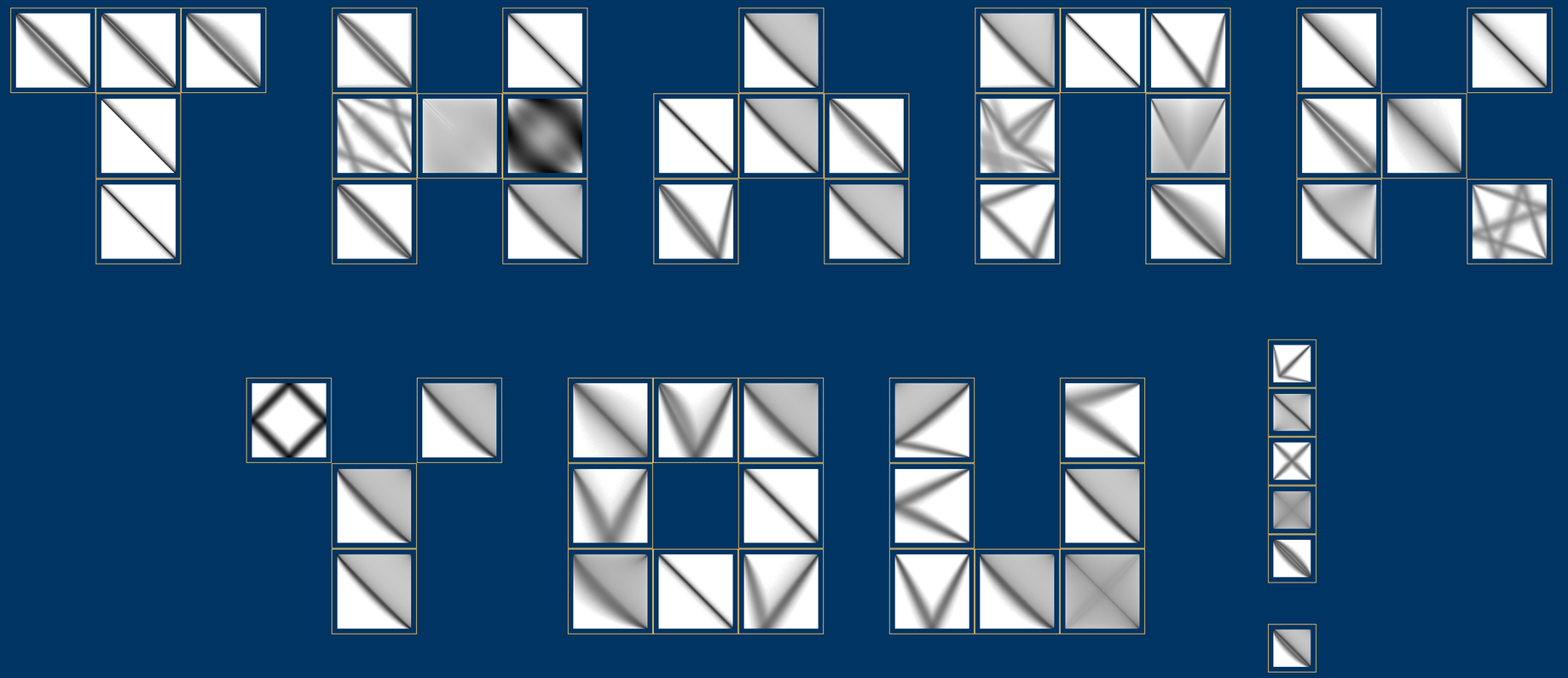
A bit optimistic, but progress continues!


Thanks
you!