A difficult prime series problem

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A challenge from Neil Sloane

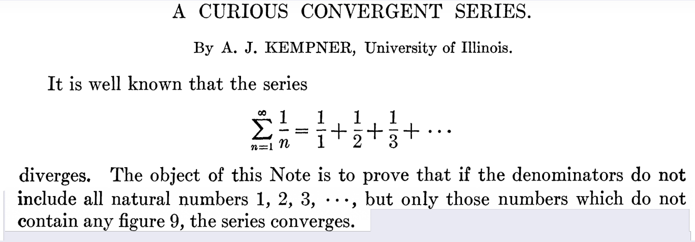
Let $V(n)$ mean: write n in base 10 but read it as if it were base 11. E.g. $V(27) = 2 * 11 + 7 = 29$. $V(n)$ is A171397. V is interesting because although the harmonic series $\sum 1/n$ diverges, it is a classic result that $S1 = \sum 1/V(n)$ converges. The decimal expansion of $S1$ is in A375805, but only 3 decimal places are known.
Second, $V(\text{prime}_n)$ is A031216. What is $S2 = \sum 1/V(\text{prime}_n)$? Its value is in A375863, but only 1 decimal place is known. Could someone calculate $S1$ and $S2$ more accurately?

As William Cheswick always says

"If brute force doesn't work, use more brute force".

History

Kempner (1914)



A CURIOUS CONVERGENT SERIES.

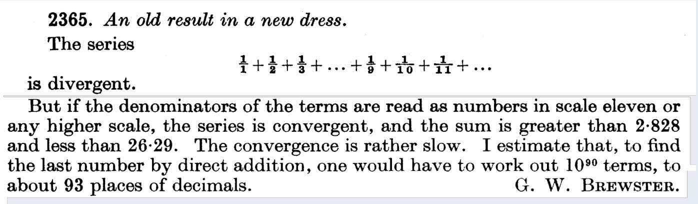
By A. J. KEMPNER, University of Illinois.

It is well known that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges. The object of this Note is to prove that if the denominators do not include all natural numbers 1, 2, 3, $\cdots$, but only those numbers which do not contain any figure 9, the series converges.

Brewster (1953)



2365. *An old result in a new dress.*

The series

$$\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} + \ldots$$

is divergent.

But if the denominators of the terms are read as numbers in scale eleven or any higher scale, the series is convergent, and the sum is greater than 2·828 and less than 26·29. The convergence is rather slow. I estimate that, to find the last number by direct addition, one would have to work out 10^{90} terms, to about 93 places of decimals.
G. W. Brewster.

A Good problem is one that teaches

- Automatic Dirichlet series [AFP00]
- Negative Binomial distribution
- Incomplete Beta Function
- Mollification of Fourier Series
- Filtering of Fourier Series
- Beurling-Selberg majorant/minorant [Mon94]
- Mellin transforms and asymptotics [Fla+94]
- Wiener's $1/f$ theorem [New75]

Themes in the solution

- Fourier Series.
- Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

- Prime zeta function

$$P(s) = \sum_{p \text{ prime}} \frac{1}{p^s}.$$

Ellipsephic Series and Automatic Dirichlet series

- After Kempner: huge literature.
- Series with digit conditions: "Ellipsephic" (due to Christian Mauduit).
- Many involving hard Analytic Number Theory.
- None (as far I could find) involve series over primes like the second series.

A few words about complexity

- What is the complexity of finding an N digit approximation to a quantity?
- Ideally it is polynomial in N (and small degree).
- In these two problems, brute force is exponential in N.
- The method presented here is also exponential, but with a much smaller base.

Summing of Series

We're summing $A = \sum_{k=1}^{\infty} a_k$. Brute force:

$$A = \underbrace{a_1 + \cdots + a_n}_{\textbf{Head}} + \underbrace{a_{n+1} + \dots}_{\textbf{Tail}}$$

- Calculate the *head* and hope that the *tail* is negligible.
- In this case, we'll see that it isn't.

The first series

- Easily handled by methods of Robert Baillie [Bai79] and Jean-François Burnol [Bur25].
- Implicitly involves finite automata and regular languages.
- Doesn't appear to apply to second sum.
- "Automaticity": how well a set of words (in our case digits) is approximated by a regular set (larger is worse).
- Thomas Dubbe [Dub25] recently proved that the set of prime digits has close to maximal automaticity.

Simple Bounds

- $$\alpha = \frac{\log 11}{\log 10} \approx 1.041392685158225$$

  .
- $10^k \leq n < 10^{k+1} \Rightarrow 11^k \leq n^\alpha < 11^{k+1}$.

Then

$$11^k = V(10^k) \leq V(n) \leq \sum_{j=0}^{k} 9 \cdot 11^j < \frac{9}{10} 11^{k+1}.$$

Thus

$$\frac{1}{11} < \frac{V(n)}{n^\alpha} < \frac{99}{10}.$$

- Write $R(n) := \frac{V(n)}{n^\alpha}$.

Why Brute force won't work (in this lifetime)

Tail Bounds

From last inequalities, for N decimal places of accuracy, for the first series, needs at least $m = c10^{N/(\alpha-1)}$ terms, where $c \approx 2.18189 \times 10^{-11}$.

N	1	2	3
$\log_{10} m$	13.497	37.6565	61.8154

For the prime series we only gain a factor of k. Still not feasible.

$R(n)$ in log scale

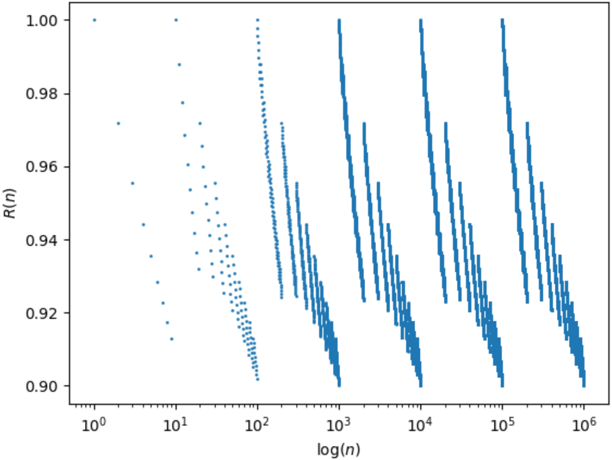


Figure: It looks periodic (almost: it is in the limit)

Strategy

- $\mathcal{P}$ is the set of primes.
- $\sum_{p \in \mathcal{P}} \frac{1}{V(p)} = \sum_{p \in \mathcal{P}} \frac{1}{R(p)p^{\alpha}}$
- Prime zeta function: $P(s) := \sum_{p \in \mathcal{P}} \frac{1}{p^s}$ for $s > 1$.
- Good algorithms exists for evaluating $P(s)$.
- Estimate the tail by using bounds for $R(n)$.
- Original bounds only buys 1 extra place of accuracy :-(.

Improved Bounds on $R(n)$

Improvement

Using Karamata's inequality [Kar32] arguments we can show: If $10^k \le n < 10^{k+1}$ then

$$1 = R(10^k) \ge R(n) \ge R(10^{k+1} - 1) > \frac{9}{10},$$

which is asymptotically tight.

An Alternate expression

A series for $V(n)$

We have

$$V(n) = \frac{1}{11} \sum_{m=-\infty}^{\infty} \lfloor 10^m n \rfloor 11^{-m}.$$

A periodic function suggested

$$F(x) = \frac{1}{11} \sum_{m=-\infty}^{\infty} \lfloor 10^{m+x} \rfloor 11^{-(m+x)}.$$

- Periodic with period 1. Let $X = \{x \in \mathbb{R} : x \equiv \log n \bmod 1, n \in \mathbb{Z}_{>0}\}$.
- $F(x)$ discontinuous for $x \in X$, continuous elsewhere.
- $R(n) = F(\log_{10} n)$.

Transferring Fourier coefficients

- $f(x)$ is periodic with period T.
- Fourier series: $f(x) \approx \sum_{k=-\infty}^{\infty} c_k e^{\frac{2\pi i k x}{T}}$.
- Dirichlet series $g(s) = \sum_{n=1}^{\infty} \frac{f(\log n)}{n^s}$
- Then $g(s) \approx \sum_{k=-\infty}^{\infty} c_k \zeta\left(s - \frac{2\pi i k}{T}\right)$.
- Dirichlet series $h(s) = \sum_{p \text{ prime}} \frac{f(\log p)}{p^s}$
- Then $h(s) \approx \sum_{k=-\infty}^{\infty} c_k P\left(s - \frac{2\pi i k}{T}\right)$.

Using Leading digits to approximate $1/F(x)$.

Leading Digits

If $a \geq 1$, say that n has leading digits a if there is a $k \geq 0$ such that

$$a \cdot 10^k \leq n < (a+1) \cdot 10^k$$

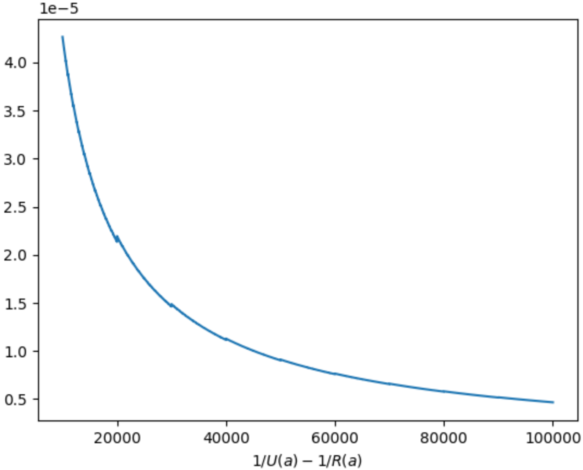
- $\mathcal{L}(a)$: integer with leading digits a.
- $\mathcal{P}_a = \mathcal{L}(a) \cap \mathcal{P}$.

Then

$$\begin{aligned} R(a) = R(a \cdot 10^k) \geq R(n) > R((a+1) \cdot 10^k - 1) \\ \geq R(a) \left(\frac{a}{a+1} \right)^{\alpha} + \frac{9}{10(a+1)^{\alpha}} := U(a), \end{aligned}$$

Good estimation

- Interval $[1/R(a), 1/U(a)]$ very small.
- When a is big this is a very good approximation.



Characteristic function for leading digits.

- Characteristic function:

$$\chi_a(x) = \begin{cases} 1 & \text{if } a \cdot 10^k \le 10^x < (a+1) \cdot 10^k \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

- $\chi_a(\log_{10} n) = 1$ if and only if $n \in \mathcal{L}(a)$.
- Periodic function of $\log_{10} n$.
- Closed form for Fourier coefficients

$$\widehat{\chi_a}(n) = \log_{10}\left(\frac{a+1}{a}\right) \operatorname{sinc}\left(\log_{10}\left(\frac{a+1}{a}\right) n\right) e^{-i\pi\left(\frac{\log(a(a+1))}{\log 10}\right) n},$$

where the normalized $\operatorname{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$ when $x \ne 0$ and 1 otherwise.

Bounds on $1/F(x)$

- Every $n \geq 10^k$ has leading digits among the set $N_k := \{10^k \leq a < 10^{k+1}\}$.
- Lower bound: $\ell_{\mathrm{k}}(\mathrm{x}) := \sum_{a \in N_k} \frac{1}{R(a)} \chi_a(x)$
- Upper bound: $\mathrm{u}_{\mathrm{k}}(\mathrm{x}) := \sum_{a \in N_k} \frac{1}{U(a)} \chi_a(x)$
- Fourier approximation:
- $\ell_k(x) \approx \sum_{j=-r}^{r} c_{k,j} \exp(2\pi ijx)$
- $u_k(x) \approx \sum_{j=-r}^{r} d_{k,j} \exp(2\pi ijx)$

Estimating the tail

$$\sum_{j=-r}^{r} c_{k,j} P^{(k+t)}\left(\alpha - \frac{2\pi i j}{\log 10}\right) \lessapprox \sum_{p\in\mathcal{P},p\geq 10^{k+t}} \frac{1}{V(p)} \lessapprox \sum_{j=-r}^{r} d_{k,j} P^{(k+t)}\left(\alpha - \frac{2\pi i j}{\log 10}\right)$$

Truncated prime zeta

$$P^{(k)}(s) := \sum_{p\in\mathcal{P},p\geq 10^k} \frac{1}{p^s}$$

The Prime Zeta Function

- Riemann zeta function and Prime zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \;\; P(s) := \sum_{p \text{ prime}} \frac{1}{p^s}, \Re s > 1.$$

- Euler Product

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

-

$$\log \zeta(s) = \sum_{p \text{ prime}} \sum_{n=1}^{\infty} \frac{1}{np^{ns}} = \sum_{n=1}^{\infty} \frac{P(ns)}{n}$$

holomorphic in $\Re s > 1$.

- Möbius inversion:

$$P(s) = \sum_{n=1}^{\infty} \mu(n) \frac{\log \zeta(ns)}{n}.$$

$\log\zeta(s)$ versus $\operatorname{Log}\zeta(s)$.

- Want to evaluate $P(s)$ for complex s.
- $\operatorname{Log}\zeta(s)$: principal value of the logarithm.
- Would like to say that $P(s) = \sum_{n=1}^{\infty} \mu(n) \frac{\operatorname{Log}\zeta(ns)}{n}$.
- When is $\log\zeta(s) = \operatorname{Log}\zeta(s)$?
- It still holds ([van83], [de 12]) when $\Re s > \sigma_- \approx 1.033908072$. Note that $\alpha > \sigma_-$.

Here σ_- is the unique solution, for $\sigma > 1$ to

$$\sum_{p \text{ prime}} \arcsin(p^{-\sigma}) = \pi.$$

Significant figures

Using leading digits between 10^4 and 10^5 and 50 terms of the Fourier series yields the value in the interval $(3.0878725218714846, 3.0878855128344616)$:

$$3.0878\ldots$$

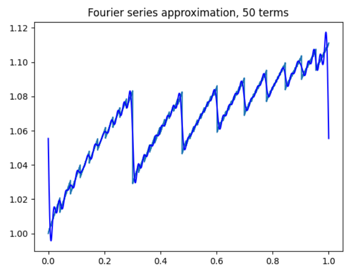


Figure: Fourier series $1/R(a)$: 50 terms

Further work

"Il n'y a pas de problèmes résolus ; il y a seulement des problèmes plus ou moins résolus."

"There are no solved problems; there are only problems that are more or less solved." — Henri Poincaré

To Do

- The Fourier series approximation is fairly weak because of jumps in the function (this is inherent).
- Are there more efficient (i.e. non-exponential) algorithms to calculate an N-digit approximation?
- In particular, can we adapt the idea of integrating against the invariant measure (as did Burnol)?
- Or is the problem of approximating the second sum inherently hard?

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Karamata's Inequality

Definition (Majorization)

Given two nonincreasing sequences, x and y of length n, say that x majorizes y if $\sigma_j(x) \geq \sigma_j(y)$ for $1 \leq j < n$ and $\sigma_n(x) = \sigma_n(y)$, where $\sigma_j(x) := \sum_{i=1}^{j} x_j$. Denote this by $x \succ y$

Theorem (Karamata's inequality)

If f is a convex function and sequence $x \succ y$, then $\sigma_n(f(x)) \geq \sigma_n(f(y))$.

Suggested by ChatGpt. But it got the sense of the inequality wrong!

Tight Inequalities

Sequences

- $n = \sum_{i=0}^{k-1} d_i \cdot 10^i$, with $0 \le d_i < 10$ and $d_{k-1} > 0$.

$$\mathcal{S}(n) := \underbrace{\frac{10^{k-1}}{n}}_{d_{k-1}\,copies} \cdots \underbrace{\frac{10^0}{n}}_{d_0\,copies}$$

- $f(x) = x^\alpha$ - convex.
- $\sigma(f(\mathcal{S}(n))) = R(n)$.
- $\mathcal{S}(a \cdot 10^k)$ (padded with 0's) majorizes $\mathcal{S}(a \cdot 10^k + d)$, $0 \le d < 10^k$: easy.
- $\mathcal{S}(a \cdot 10^k + d)$ majorizes $\mathcal{S}((a+1) \cdot 10^k - 1)$: ugly.

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