

# On a Generalized Moment Integral containing Riemann's Zeta Function: Analysis and Experiment

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$$\sigma, r \in \mathbb{R}, \quad \sigma > 1, r > 0$$

$$Z(\sigma, r) \equiv \int_{-\infty}^{\infty} \frac{\zeta(\sigma \pm it) r^{\sigma \pm it}}{(\sigma \pm it)} dt = -i \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\zeta(v) r^v}{v} dv, \quad (1)$$

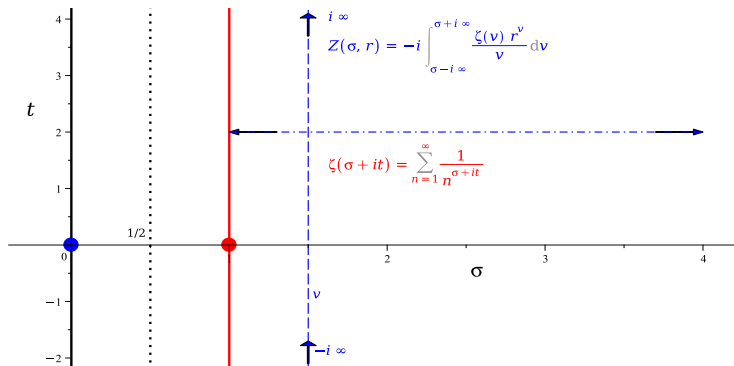
1. <https://doi.org/10.64700/mmm.48> (Generalized Moment Integral...)
2. <https://doi.org/10.48550/arXiv.1308.6361> (A remarkable Definite Integral)
3. [dx.doi.org/10.7153/jca-2024-24-03](https://doi.org/10.7153/jca-2024-24-03) (An extension...)
4. <https://arxiv.org/abs/2107.12559> (Determining the Indeterminate...)

# Summary

- refresher
  - zeta
  - analytic continuation
  - discontinuities
- background
- strategy
- derivations
  - enigma
- graphic examples
- Cesàro summation
- numerical tests
- digression
  - offsets
  - correlation
- My questions
- Q&A

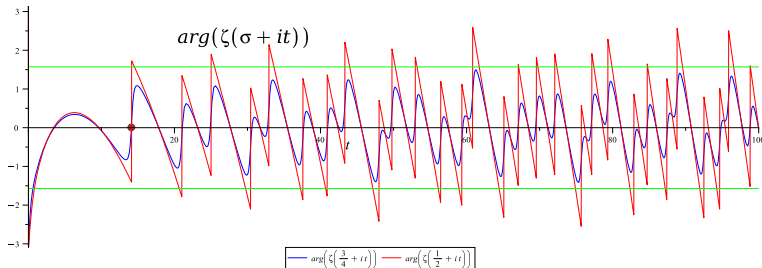
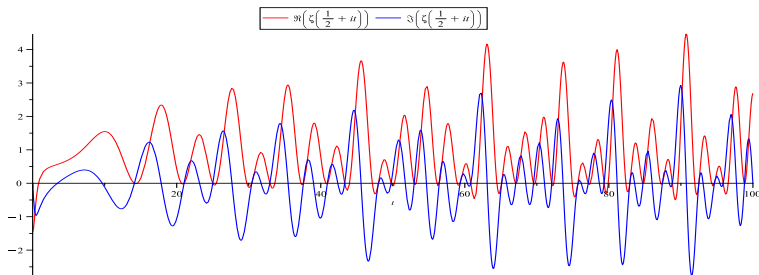
$$s = \sigma + it, \quad \zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \sigma > 1$$

We are interested in  $\sigma < 1$ , using both *analytic continuation* and *numerical evaluation* (Mathematica and Maple).



$$\zeta(1-s) = 2\Gamma(s) \cos(\pi s/2) (2\pi)^{-s} \zeta(s),$$

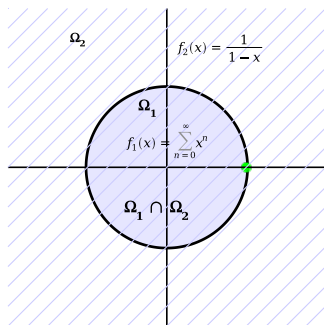
# $\zeta(\sigma + it)$ by the numbers



# Analytic Continuation - a Review

*Definition: Let  $f_1$  and  $f_2$  be analytic functions on domains  $\Omega_1$  and  $\Omega_2$  respectively and suppose that the intersection  $\Omega_1 \cap \Omega_2$  is not empty and that  $f_1 = f_2$  on  $\Omega_1 \cap \Omega_2$ . Then  $f_2$  is an analytic continuation of  $f_1$  on  $\Omega_2$  and vice versa. Moreover, if it exists, the analytic continuation of  $f_1$  to  $\Omega_2$  is unique.*

Example:



$f_1(x) = f_2(x)$  in the overlap region  $|x| < 1$ . Since  $f_2(x)|_{x=2} = -1$ , then “*the analytic continuation of  $f_1(x) = -1$  at  $x = 2$* ”. It does NOT mean that  $\sum_{n=0}^{\infty} 2^n = -1$ . A more formal way is to say that there exists a “parent” function  $F(x)$  with two different representations  $f_1(x)$  and  $f_2(x)$  each in their own domain of validity. But writing  $F(2) = -1$  doesn't convey much information

# Definitions

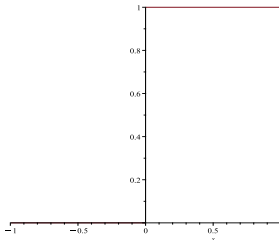
## Definition (The modified “floor” function)

$\lfloor x \rfloor$  means the greatest integer less than **but not equal to**  $x$ ,  $x \neq n$ .  
Traditionally,  $\lfloor n \rfloor = n$ . Here,  $\lfloor n \rfloor$  is undefined.

## Definition (The Heaviside step function)

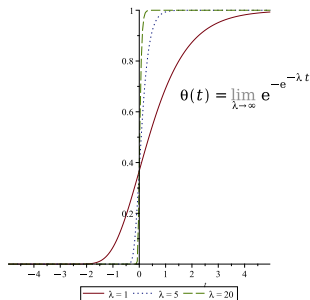
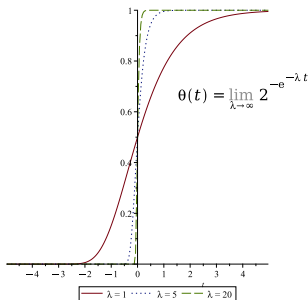
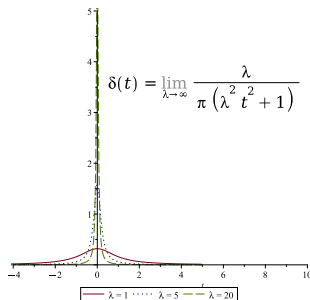
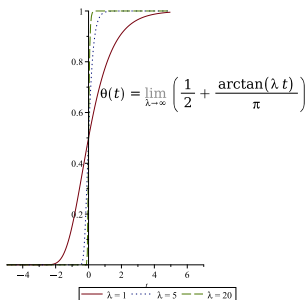
$$\theta(x) = \begin{cases} 0 & x < 0 \\ \text{undefined} & x = 0 \\ 1 & x > 0 \end{cases}$$

$\theta(0) \equiv 1/2$  (*signal processing*),  $\theta(0) \equiv 1$  (*digital engineering*),  
 $\theta(0) \equiv 0$  (*probability and statistics*)



The Dirac delta function:  $\delta(t) = \frac{d}{dt} \theta(t)$

# Numerical approximations:



# At a pole: directed vs. directional limits

Consider

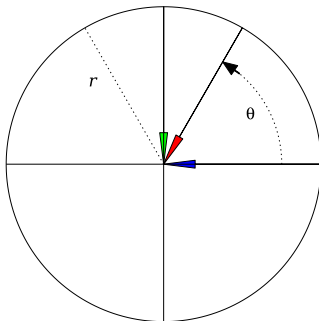
$$F(z) = 1/z$$

Let

$$z = r e^{i\theta}$$

$$F(z) = F_R(r, \theta) + iF_I(r, \theta) = \frac{\cos(\theta)}{r} - i \frac{\sin(\theta)}{r}$$

What is the value of  $F_R(0, \theta)$  and  $F_I(0, \theta)$  treated as independent functions?



As  $r \rightarrow 0$ ,  $F_R(r, \theta) \rightarrow \pm\infty$  except if  $\theta = \pm\pi/2$  in which case  $F_R(r, \theta) = 0$

As  $r \rightarrow 0$ ,  $F_I(r, \theta) \rightarrow \pm\infty$  except if  $\theta = \{0, \pi\}$  in which case  $F_I(r, \theta) = 0$

# Background and Motivation

Theorem (Glasser's "remarkable integral" (originally  $b = 1$ ))

If  $F(v)$  is meromorphic in  $\mathfrak{S}_b \equiv -b < \Re(v) < 0$ , **the integral exists** and  $F(v) + F(-ib - v) = 0$ , then

$$\int_{-\infty}^{\infty} F(v) dv = -\pi i \sum_j \operatorname{Res} (F(v_j)) |_{v_j \in \mathfrak{S}_b}$$

CAVEAT: points at infinity do not contribute.

From the theorem and use of Mellin transforms we find:

$$\int_0^{\infty} \frac{\zeta_R\left(\frac{1}{2} + iv\right)}{\cosh(\pi v)} dv = \gamma - 1$$

$$\int_{-\infty}^{\infty} \left( \frac{\zeta(1/2 + iv)}{1/2 + iv} - \frac{\zeta(3/2 - iv)}{3/2 - iv} \right) dv = -2\pi$$

$$\int_{-\infty}^{\infty} \frac{\zeta(1/2 + iv)}{1/2 + iv} dv = -\pi$$

$$\int_0^{\infty} \zeta_R(1/2 + iv) dv = -\pi$$

and the parent function

$$Z(\sigma, r) \equiv \int_{-\infty}^{\infty} \frac{\zeta(\sigma + it) r^{\sigma + it}}{\sigma + it} dt$$

1. Find a representation of  $Z(\sigma, r)$  when  $\sigma > 1$ ;

$$\int_{-\infty}^{\infty} \frac{\zeta(\sigma + it) r^{\sigma + it}}{\sigma + it} dt = Z(\sigma, r) = Z_R(\sigma, r) + i Z_I(\sigma, r)$$

2. Differentiate wrt  $r$  to yield

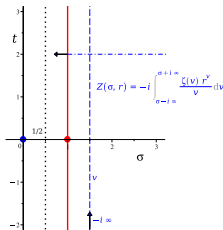
$$\int_{-\infty}^{\infty} \zeta(\sigma + it) r^{\sigma + it} dt = r \frac{\partial}{\partial r} Z(\sigma, r), \quad \sigma > 1$$

3. Analytically continue  $r \frac{\partial}{\partial r} Z(\sigma, r)$  to  $0 < \sigma \leq 1$  by evaluating the residue at the point  $\sigma = 1$ ,  $t = 0$ , i.e.:

$$\int_{-\infty}^{\infty} \zeta(\sigma + it) r^{\sigma+it} dt = r \frac{\partial}{\partial r} Z(\sigma, r) - 2\pi r, \quad 0 < \sigma < 1, \quad \sigma = 1$$

4. Set  $r = 1$ :  $Z(\sigma, 1)|_{\sigma < 1} = \int_{-\infty}^{\infty} \zeta(\sigma + it) dt = \lim_{r \rightarrow 1} \left( r \frac{\partial}{\partial r} Z(\sigma, r) - 2\pi r \right) \quad 0 < \sigma < 1, \quad \sigma = 1$

5. At any point, simplify, for example:  $\int_0^\infty \frac{\zeta(\sigma+it)r^{\sigma+it}}{\sigma+it} dt = \frac{1}{2} Z_R(\sigma, r).$



# Divergent integrals - application

When the left-hand side is divergent:

- from Quantum Electrodynamics :

$$\int_0^{\infty} f(x) dx = \lim_{T \rightarrow \infty} \int_0^T f(x) dx ; \quad (2)$$

- Principal Value Regularization with  $a \in \mathbb{R}$ ,  $a > 0$  :

$$\int_0^{\infty} f(x)/(x-a) dx = \lim_{\epsilon \rightarrow 0} \int_0^{\infty} f(x)/(x-a \pm i\epsilon) dx ; \quad (3)$$

- Analytic Regularization where  $\Re(a) > 0$ ,  $\nu \in \mathbb{R}$ :

$$\int_0^{\infty} f(x)/(x+a)^n dx = \lim_{\nu \rightarrow n} \int_0^{\infty} f(x)/(x+a)^{\nu} dx ; \quad (4)$$

- and Dimensional Regularization:

$$\int_0^{\infty} f(x) d^n x = \lim_{\epsilon \rightarrow 0} \int_0^{\infty} f(x) d^{n+\epsilon} x . \quad (5)$$

Typically leads to a pole of form  $1/\epsilon$  etc.

# Derivation

$$\zeta(\sigma + it) = \sum_{j=1}^{\infty} \frac{1}{j^{\sigma}} e^{-it \log(j)}, \quad \sigma > 1$$

$$Z(\sigma, r) = \int_{-\infty}^{\infty} \frac{\zeta(\sigma + it) r^{\sigma + it}}{\sigma + it} dt = r^{\sigma} \sum_{j=1}^{\infty} \frac{1}{j^{\sigma}} \int_{-\infty}^{\infty} \left( \frac{\sigma \cos(t \ln(\frac{r}{j}))}{\sigma^2 + t^2} + \frac{t \sin(t \ln(\frac{r}{j}))}{\sigma^2 + t^2} \right) dt$$

and the imaginary component vanishes by anti-symmetry.

Lemma

$$\int_{-\infty}^{\infty} \frac{t \sin(at)}{\sigma^2 + t^2} dt = \pi e^{-a\sigma}, \quad a > 0$$

$$\int_{-\infty}^{\infty} \frac{\cos(at)}{\sigma^2 + t^2} dt = \frac{\pi}{\sigma} e^{-a\sigma}, \quad a > 0$$

with special cases when  $a = 0$  and  $a < 0$ .

Integrating each term and employing the lemmas gives

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\sin(t \ln(r/j))}{\sigma^2 + t^2} t dt &= \pi(r/j)^{-\sigma}, & j < r, \\ &= -\pi(r/j)^{\sigma}, & j > r, \\ &= 0 & j = r. \end{aligned}$$

$$\begin{aligned} \sigma \int_{-\infty}^{\infty} \frac{\cos(t \ln(r/j))}{\sigma^2 + t^2} dt &= \pi(r/j)^{-\sigma}, & j < r, \\ &= \pi(r/j)^{\sigma}, & j > r, \\ &= \pi & j = r. \end{aligned}$$

# Derivation continued

Case 1:  $r \neq n$ :

$$\begin{aligned}
 Z(\sigma, r) &= \pi \sum_{j=1}^{\lfloor r \rfloor} \left(\frac{j}{r}\right)^{-\sigma} \left( \left(\frac{j}{r}\right)^{\sigma} + \left(\frac{j}{r}\right)^{\sigma} \right) + \pi \sum_{j=\lfloor r \rfloor+1}^{\infty} \frac{\left( \left(\frac{j}{r}\right)^{-\sigma} - \left(\frac{j}{r}\right)^{-\sigma} \right)}{\left(\frac{j}{r}\right)^{\sigma}} \\
 &= \pi \sum_{j=1}^{\lfloor r \rfloor} 2 + 0 = 2\pi \lfloor r \rfloor, \quad r \neq n, \sigma > 1.
 \end{aligned}$$

Case 2:  $r = n$ :

$$\begin{aligned}
 Z(\sigma, n) &= \pi \sum_{j=1}^{n-1} \left(\frac{j}{n}\right)^{-\sigma} \left( \left(\frac{j}{n}\right)^{\sigma} + \left(\frac{j}{n}\right)^{\sigma} \right) + \pi + \pi \sum_{j=n+1}^{\infty} \frac{\left( \left(\frac{j}{n}\right)^{-\sigma} - \left(\frac{j}{n}\right)^{-\sigma} \right)}{\left(\frac{j}{n}\right)^{\sigma}} \\
 &= \pi \sum_{j=1}^{n-1} 2 + \pi + 0 = 2\pi(n-1) + \pi = 2\pi(n-1/2), \quad \sigma > 1.
 \end{aligned}$$

i.e.

$$\begin{aligned}
 Z(\sigma, r) &\equiv \int_{-\infty}^{\infty} \frac{\zeta(\sigma + it) r^{\sigma + it}}{\sigma + it} dt = 2\pi \lfloor r \rfloor & r \neq n, \sigma > 1 \\
 &= 2\pi(n-1/2) & r = n, \sigma > 1
 \end{aligned}$$

# An enigmatic derivation courtesy of Christophe Vignat (Tulane University)

From "Theory of the Riemann Zeta Function", Titchmarsh and Heath-Brown",  
(Eq. 2.1.4) - This is related to the Mellin transform of  $Z(\sigma, r)$ :

$$\zeta(s) = s \int_1^\infty \frac{[x] - x + 1/2}{x^{s+1}} dx + \frac{1}{2} + \frac{1}{s-1}, \quad \sigma > 0,$$

$$\begin{aligned} \frac{\zeta\left(\frac{3}{2} - it\right)}{\frac{3}{2} - it} &= \int_1^\infty \frac{[x] - x + \frac{1}{2}}{x^{\frac{5}{2} - it}} dx + \frac{1}{\left(\frac{3}{2} - it\right)\left(\frac{1}{2} - it\right)} + \frac{1}{3 - 2it} \\ &= \int_1^\infty \frac{[x] - x + \frac{1}{2}}{x^{\frac{5}{2} - it}} dx + \frac{\frac{1}{2} + it}{t^2 + 1/4} - \frac{\frac{3}{2} + it}{2(t^2 + 9/4)}. \end{aligned}$$

First, pre-multiply by  $r^{-it}$  and integrate

$$\begin{aligned} \int_{-\infty}^\infty \frac{\zeta\left(\frac{3}{2} - it\right) r^{-it}}{\frac{3}{2} - it} dt &= \int_1^\infty \frac{([x] - x + \frac{1}{2})}{x^{\frac{5}{2}}} \int_{-\infty}^\infty \left(\frac{x}{r}\right)^{it} dt dx \\ &\quad + \int_{-\infty}^\infty \frac{r^{-it}(1/2 + it)}{t^2 + 1/4} dt - \frac{1}{2} \int_{-\infty}^\infty \frac{r^{-it}(3/2 + it)}{t^2 + 9/4} dt \end{aligned}$$

where the double integral operators in the first (right-hand side) term have been transposed.

$$\begin{aligned} \int_{-\infty}^\infty \frac{r^{-it}\left(\frac{1}{2} + it\right)}{t^2 + \frac{1}{4}} dt &= \frac{1}{2} \int_{-\infty}^\infty \frac{\cos(t \ln(r)) + 2t \sin(t \ln(r))}{t^2 + 1/4} dt \\ &= \frac{2\pi}{\sqrt{r}} && \text{if } r > 1; \\ &= 2\pi && \text{if } r = 1; \\ &= 0 && \text{if } 0 < r < 1 \end{aligned}$$

# Enigma cont'd

$$\begin{aligned}
 \frac{1}{2} \int_{-\infty}^{\infty} \frac{r^{-it} \left( \frac{3}{2} + it \right)}{t^2 + \frac{9}{4}} dt &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\frac{3}{2} \cos(t \ln(r)) + t \sin(t \ln(r))}{t^2 + 9/4} dt \\
 &= \frac{\pi}{r^{\frac{3}{2}}} && \text{if } r > 1; \\
 &= \pi && \text{if } r = 1; \\
 &= 0 && \text{if } 0 < r < 1.
 \end{aligned}$$

From both Maple and Mathematica:

$$\int_{-\infty}^{\infty} \cos\left(t \ln\left(\frac{x}{r}\right)\right) dt = 2\pi \delta\left(\ln\left(\frac{x}{r}\right)\right)$$

along with the change of variables  $\ln\left(\frac{x}{r}\right) := x$ , we obtain

$$\begin{aligned}
 &\int_1^{\infty} x^{-\frac{5}{2}} \left( \lfloor x \rfloor - x + \frac{1}{2} \right) \int_{-\infty}^{\infty} \left( \frac{x}{r} \right)^{it} dt dx \\
 &= \int_1^{\infty} x^{-\frac{5}{2}} \left( \lfloor x \rfloor - x + \frac{1}{2} \right) \int_{-\infty}^{\infty} \cos\left(t \ln\left(\frac{x}{r}\right)\right) dt dx \\
 &= -\pi r^{-\frac{3}{2}} \int_{-\ln(r)}^{\infty} (2r e^x - 2 \lfloor r e^x \rfloor - 1) e^{-3x/2} \delta(x) dx \\
 &= -\pi r^{-\frac{3}{2}} (2r - 2 \lfloor r \rfloor - 1) && \text{if } r > 1 \\
 &= \pi n^{-\frac{3}{2}} && \text{if } r = n, \ n \geq 1; \\
 &= 0 && \text{if } 0 < r < 1.
 \end{aligned}$$

# Enigma cont'd

Putting it all together, this derivation yields the prospective, but aberrant, identity

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\zeta\left(\frac{3}{2} + it\right)}{\frac{3}{2} + it} r^{\frac{3}{2} + it} dt \\&= 2\pi \lfloor r \rfloor && \text{if } r > 1; \\&= 2\pi n && \text{if } r = n, n \geq 1; \\&= 0 && \text{if } 0 < r < 1.\end{aligned}$$

This compares to the previous derivation (with a different definition of  $\lfloor r \rfloor \rightarrow |r|$ )

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\zeta\left(\frac{3}{2} + it\right)}{\frac{3}{2} + it} r^{\frac{3}{2} + it} dt \\&= 2\pi |r| && \text{if } r > 1; \\&= 2\pi (n - 1/2) && \text{if } r = n, n \geq 1; \\&= 0 && \text{if } 0 < r < 1.\end{aligned}$$

## Remark

Nowhere in this derivation does  $r = n$  suggest the existence of a special value at the discontinuity, and, the case  $r = n$  simply corresponds to the reduction  $r \rightarrow n$  of the more general case corresponding to  $r > 1$  because  $\lfloor n \rfloor = n$ .

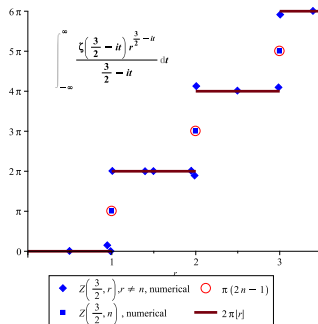
## Remark

$$\text{hint : } Z(\sigma, n) = \int_{-\infty}^{\infty} \lim_{r \rightarrow n} \frac{\zeta(\sigma + it) r^{\sigma + it}}{\sigma + it} dt \neq \lim_{r \rightarrow n} \int_{-\infty}^{\infty} \frac{\zeta(\sigma + it) r^{\sigma + it}}{\sigma + it} dt$$

# Example: $\sigma = 3/2$

$$Z(\sigma, r) = 2\pi \lfloor r \rfloor, \quad r \neq n, \sigma > 1$$

$$= 2\pi(n - 1/2) \quad r = n, \sigma > 1$$



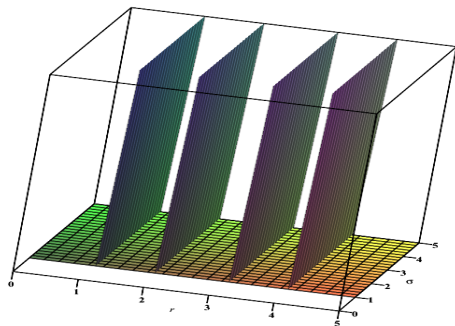
The staircase function  $Z(\frac{3}{2}, r)$  over a small range of  $r$ , compared analytically and numerically. Note that the left and right limits of each of the horizontal “treads” are open and that the values at the midpoints of the “risers” are obtained by both an analytic and a numerical evaluation of the integral, not by decree.

## Remark

$$Z'(\sigma, r) \equiv r \frac{\partial}{\partial r} Z(\sigma, r) = \int_{-\infty}^{\infty} \zeta(\sigma + it) r^{\sigma + it} dt = 2\pi r \delta(r - n), \quad \sigma > 1$$

$$Z(\sigma, r + m) = Z(\sigma, r) + 2\pi m$$

# The Dirac comb function



The Dirac comb function for  $\sigma > 1$ :

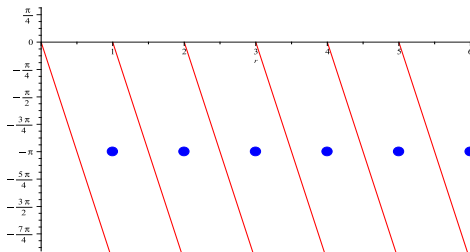
$$Z'(\sigma, r) \equiv r \frac{\partial}{\partial r} Z(\sigma, r) = \int_{-\infty}^{\infty} \zeta(\sigma + it) r^{\sigma+it} dt = 2\pi r \delta(r - n).$$

# Analytic Continuation to: $0 < \sigma < 1$

We analytically continue  $Z(\sigma, r)$  to  $0 < \sigma < 1$  by subtracting the residue at  $\sigma = 1$ , yielding

$$Z(\sigma, r)|_{0 < \sigma < 1} \equiv \int_{-\infty}^{\infty} \frac{\zeta(\sigma + it) r^{\sigma + it}}{\sigma + it} dt = 2\pi (|r| - r), \quad r \neq n,$$

$$= -\pi, \quad r = n.$$



The function  $2\pi (|r| - r)$  over a small range of  $r$  with  $\sigma = 1/2$ , demonstrating invariance if  $r \rightarrow r + m$ ; therefore  $Z(\sigma, r + m) = Z(\sigma, r)$  if  $0 < \sigma < 1$ .

Differentiating once:

$$Z'(\sigma, r) \equiv r \frac{\partial}{\partial r} Z(\sigma, r)|_{0 < \sigma < 1} = \int_{-\infty}^{\infty} \zeta(\sigma + it) r^{\sigma + it} dt = -2\pi r, \quad r \neq n$$

$$= \infty \quad ???, \quad r = n.$$

$$\sigma = 1$$

What if  $\sigma = 1$ ?

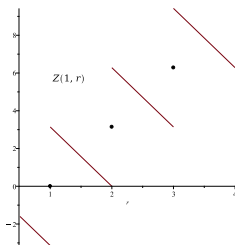
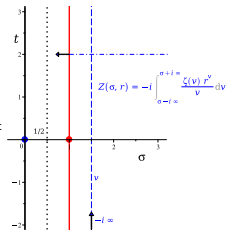
$$\zeta(s) \sim 1/(s-1) + \gamma - \gamma(1)(s-1) + O((s-1)^2)$$

Let  $s = \sigma + it$ , then if  $\sigma = 1$ ,

$$\zeta(1+it) \sim -i/t + \gamma - \gamma(1)it + O(t^2)$$

and  $Z_I(1, r)$  contains the singularity; Therefore, by deforming the contour and adding half the residue we get

$$Z(\sigma, r)|_{\sigma=1} \equiv \int_{-\infty}^{\infty} \frac{\zeta(1+it) r^{1+it}}{1+it} dt = \pi(2\lfloor r \rfloor - r), \quad r \neq n, \\ = \pi(n-1), \quad r = n.$$



Differentiating once wrt  $r$  yields

$$\int_0^\infty \zeta(1+it) r^{1+it} dt = -\pi r/2, \quad r \neq n,$$

$$= \infty \text{ ???}, \quad r = n.$$

# Cesàro Summation

$$H(T) = \int_0^T h(t) dt$$

Subdivide the integral into small parts defined by

$$h(t_j) = \int_{t_j}^{t_j+\delta} h(t) dt$$

so that

$$H(t_m) = \sum_{j=1}^m h(t_j)$$

where  $t_1 = 0$ ,  $t_J + \delta = T$ . Then  $H(T) = H(t_J)$  and we use  $\delta = 1$  and  $J \sim 4000$ . Form the partial sums

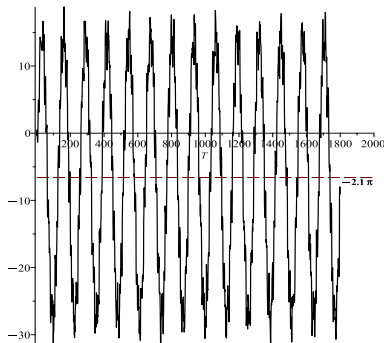
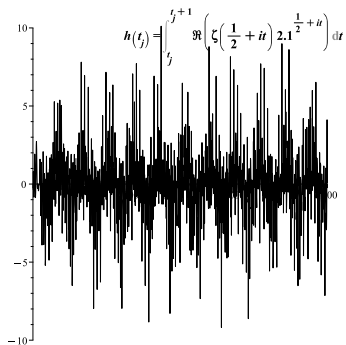
$$P_k = \sum_{m=1}^k H(t_m), \quad k \leq J.$$

Then  $H(T)$  equals the running average of each partial sum if the sum is convergent, i.e.

$$H_n = \frac{1}{n} \sum_{k=1}^n P_k \quad n \leq J,$$

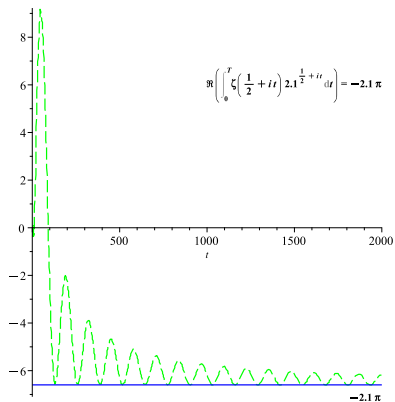
and, regularizes the sum if it is divergent.

# Example $\sigma = \frac{1}{2}$ , $r = 2.1$



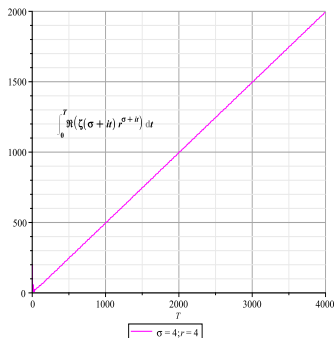
The corresponding partial sums  $P_k$ , each of which also equates to the numerical value of the integral as a function of its upper limit  $T$ . The asymptotic limit  $2.1\pi$  is also shown.

# Surprise!

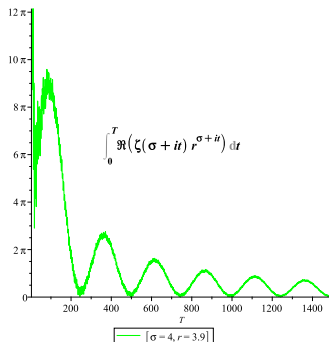


Running average of the partial sums with  $\sigma = 1/2$ ,  $r = 2.1$

# Back to the region $\sigma > 1$



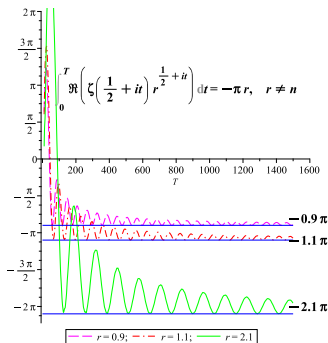
Cesàro approximation to the function  $Z'(\sigma, r) = 2\pi r \delta(r - n)$  with  $\sigma = r = 4$ , suggesting divergence of order  $T/2$ .



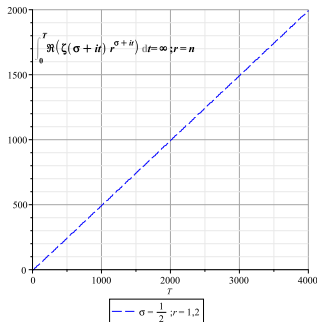
Cesàro approximation to the function  $Z'(\sigma, r) = 2\pi r \delta(r - n)$  with  $\sigma = 4, r = 3.9$ , suggesting convergence to zero.

Numerical approximations to  $Z'(\sigma, r)$  prior to analytic continuation.

# Various values of $r$

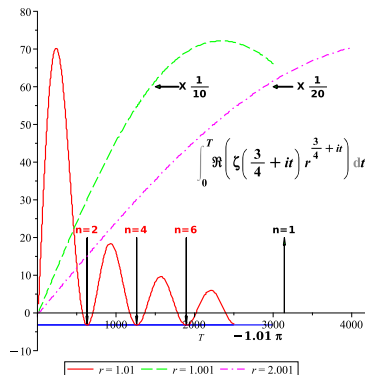


Cesàro approximation to the function  $Z'(\sigma, r)$  with  $\sigma = 1/2$ ,  $r = 0.9$ ,  $r = 1.1$  and  $r = 2.1$ , suggesting convergence to  $-\pi r$  as  $T$  increases.



Cesàro approximation to the function  $Z'(\sigma, r)$  with  $\sigma = 1/2$ ,  $r = 1$  and  $r = 2$ , revealing a reasonable indication of divergence. The two cases are numerically indistinguishable within graphical resolution.

# The case $r \sim 1$ , $0 < \sigma < 1$



Study with  $r$  close to 1. Down pointing vertical arrows indicate multiples of the solution points  $t$  where  $t \ln(1.01) = n\pi$ ,  $n \geq 1$ . Up pointing arrow labels two cases when  $r \approx 1$ .

## Remark

Note:  $r^{3/4+it} = r^{3/4} (\cos(t \ln(r)) + i \sin(t \ln(r)))$

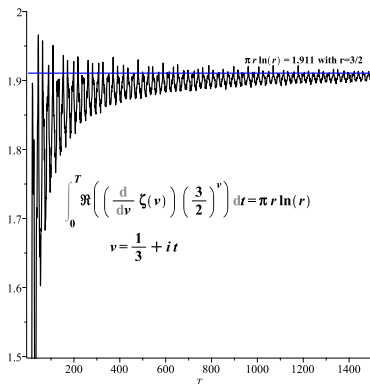
# Testing derivatives wrt $\sigma$

Recall:

$$Z(\sigma, r)|_{0 < \sigma < 1} \equiv \int_{-\infty}^{\infty} \frac{\zeta(\sigma + i t) r^{\sigma + i t}}{\sigma + i t} dt = 2\pi (|r| - r), \quad r \neq n,$$

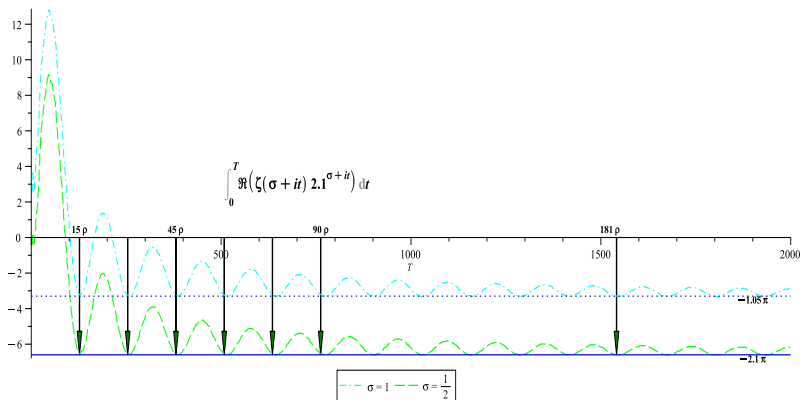
$$= -\pi, \quad r = n.$$

$$\frac{\partial^2}{\partial r \partial \sigma} Z(\sigma, r) = 2\pi r \log(r), \quad \sigma < 1, \quad r \neq n, \quad \frac{\partial}{\partial \sigma} Z(\sigma, r) = 0$$



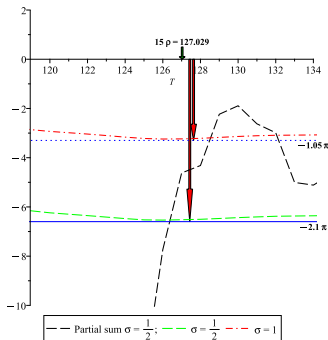
A test of the above with  $\sigma = 1/3$ ,  $r = 3/2$ .

# One value of $r$ and various $\sigma$

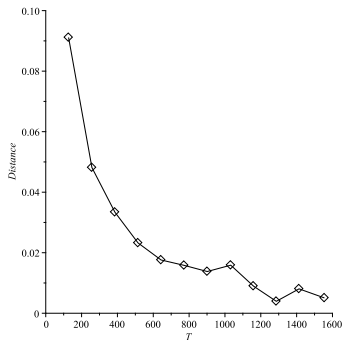


Cesàro approximations to the function  $Z'(\sigma, r)$  with  $r = 2.1$  and  $0 < \sigma < 1$  compared with  $\sigma = 1$ . Various choices using  $\sigma = 1/3$ ,  $\sigma = 3/4$  and  $\sigma = 7/8$  are not shown because they are numerically indistinguishable, within the resolution of the figure. The case  $\sigma = 1/2$  was selected. The arrows mark integral multiples of a conjectured period  $\rho = 2\pi / \log(r)$ .

# Convergence?

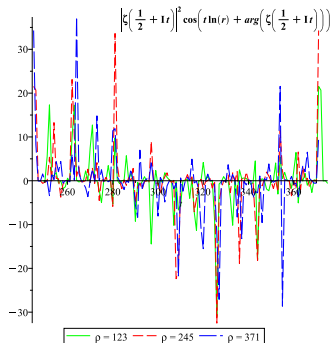


The small down arrow marks the point corresponding to the fifteenth harmonic of the assumed period  $\rho$ . The two other arrows indicate the minimum of the respective Cesàro approximation when  $\sigma = 1/2$  and  $\sigma = 1$ . The partial sum corresponding to the numerical integration is also shown crossing the line ( $-2.1\pi$ ) that marks the asymptotic limit.

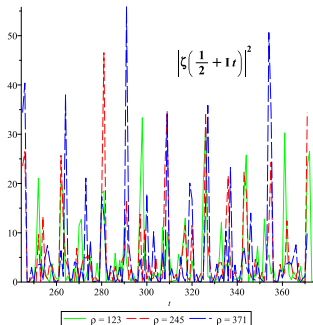


For each minimum marked in the previous figure, this shows the distance between the Cesàro estimate and the asymptotic line  $-2.1\pi$ .

# A leap of conjecture



This shows overlain segments of the argument of  $Z'(1/2, r)$ , each shifted relative to  $t = 0$  by the amounts  $\rho$  indicated. In all cases,  $r = 2.1$ .



This shows overlain segments of the function  $|\zeta(1/2 + it)|^2$ , each shifted relative to  $t = 0$  by the amounts  $\rho$  indicated.

This Figure overlays two quantities, each shifted by  $t \rightarrow t + \rho$ .

# A quantitative test

For any two continuous functions  $f(t)$  and  $g(t)$ , we borrow from statistics, and define a correlation coefficient  $\text{Cor}(f, g)$  by

$$\text{Cor}(f, g) = \frac{\text{Cov}(f, g)}{\sqrt{\text{Var}(f) \text{Var}(g)}} \quad (4)$$

where the “Covariance” between two functions  $f(t)$  and  $g(t)$  is defined by

$$\text{Cov}(f, g) \equiv E(fg) - E(f)E(g), \quad (5)$$

the “Variance” by

$$\text{Var}(f) \equiv E(f^2) - E(f)^2 \quad (6)$$

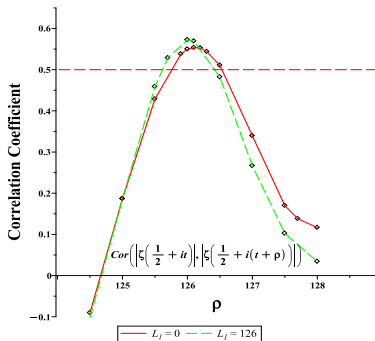
in terms of the fundamental quantity “Expectation” defined for any function  $h(t)$  by

$$E(h) \equiv \frac{1}{L_2 - L_1} \int_{L_1}^{L_2} h(t) dt. \quad (7)$$

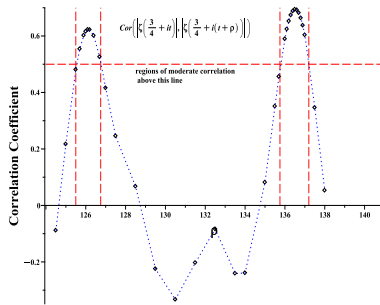
$\text{Cor}(f, g) = \pm 1$  means  $f(t)$  and  $g(t)$  are perfectly/anti correlated over the interval  $L_2 - L_1$

Otherwise the degree of correlation is in the eye of the observer.

# Statistical correlation: two examples

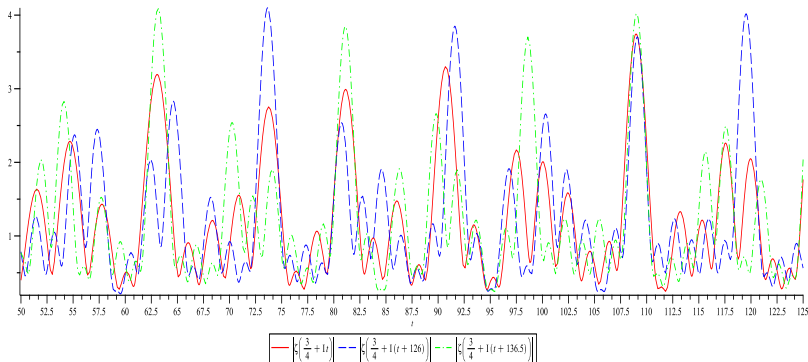


This Figure shows the correlation coefficient between  $f(t) = |\zeta(1/2 + it)|$  and  $f(t) = |\zeta(1/2 + i(t + \rho))|$  as a function of  $\rho$ , between offset intervals starting at  $t = L_1$  as shown. Values above 0.5 are considered to be moderately correlated.



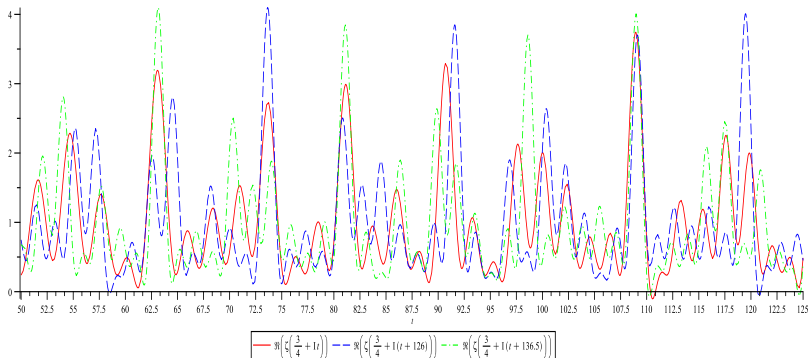
This Figure shows the correlation coefficient between  $f(t) = |\zeta(3/4 + it)|$  and  $f(t) = |\zeta(3/4 + i(t + \rho))|$  as a function of  $\rho$ .

# Three offsets

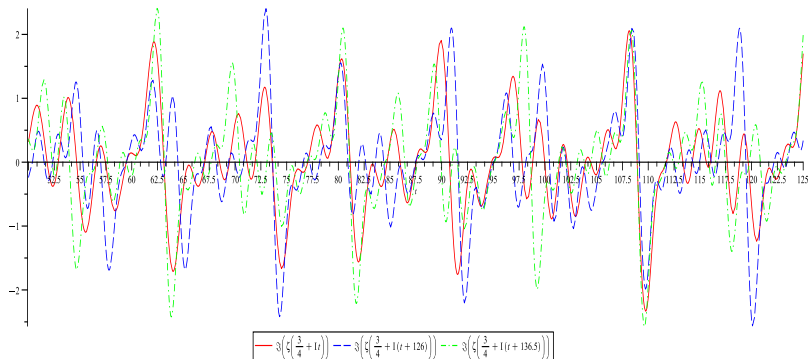


A comparison of  $|\zeta(3/4 + i(t + \rho))|$  with two different offsets over a small range of  $t$ .

The following two figures present the real and imaginary components of the functions shown in the previous figure.



A comparison of  $\Re\zeta(3/4 + i(t + \rho))$  with two different offsets over a small range of  $t$ .



A comparison of  $\Im\zeta(3/4 + i(t + \rho))$  with two different offsets over a small range of  $t$ .

# Questions?

1. What is the justification (if any) for approaching a pole in a certain direction?
2. Does  $\int_0^\infty \zeta_R(1/2 + iv)dv = -\pi$  or is it infinite? Or both?
3. Why does Cesàro summation approach its limit as it does?
4. What is the origin of the offset phase  $\rho$  and its integer multipliers?
5. Is the observed offset a real effect or an artifact of numerical algorithms?

