Pattern avoidance classes of parking permutations with fixed lucky sets

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Joint work with Melanie Ferreri (William & Mary)

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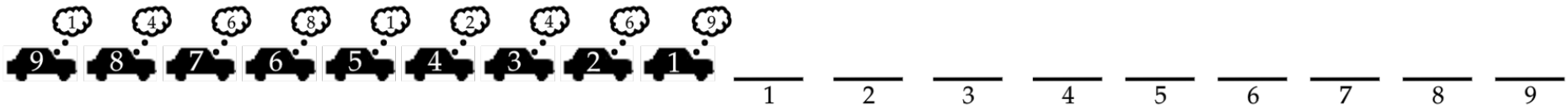
If the all the cars are able to park (under the parking rule) then $a = (a_1, a_2, \ldots, a_n)$ is called a parking function.

Parking functions

Is $\alpha = (9, 6, 4, 2, 1, 8, 6, 4, 1)$ a parking function?

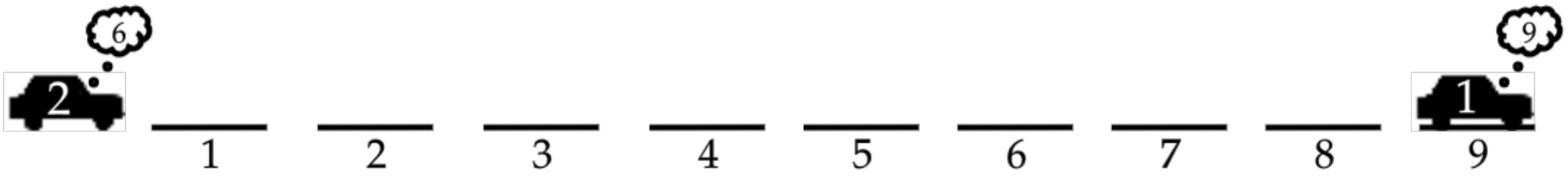
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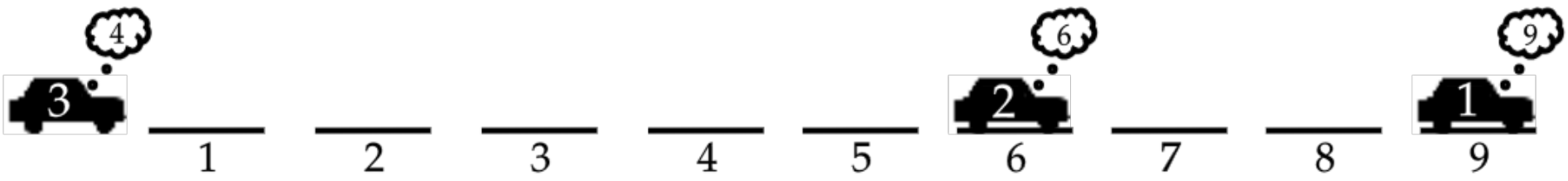
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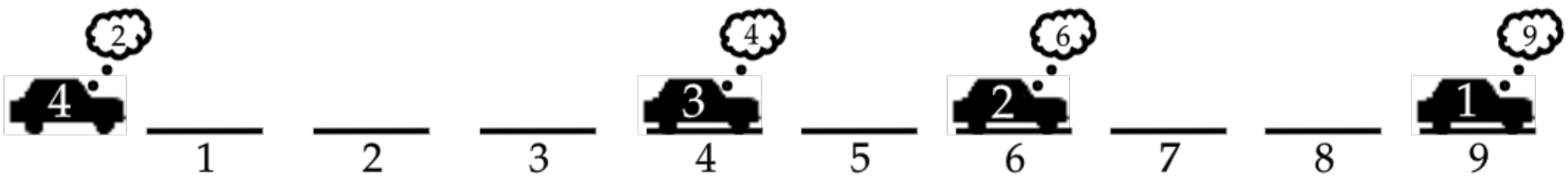
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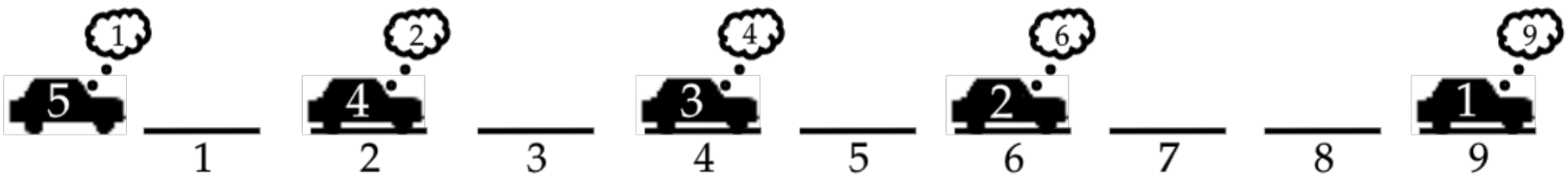
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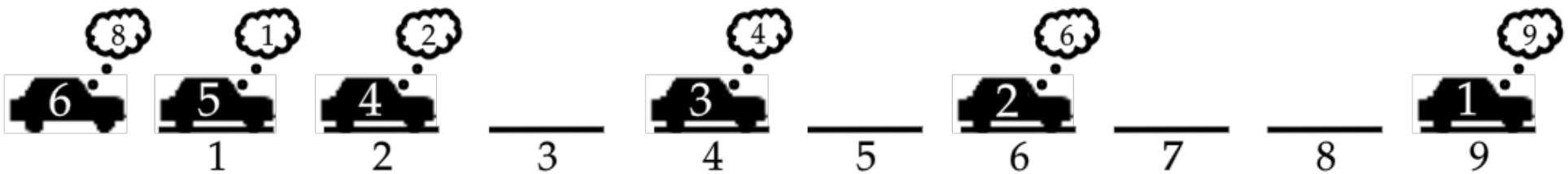
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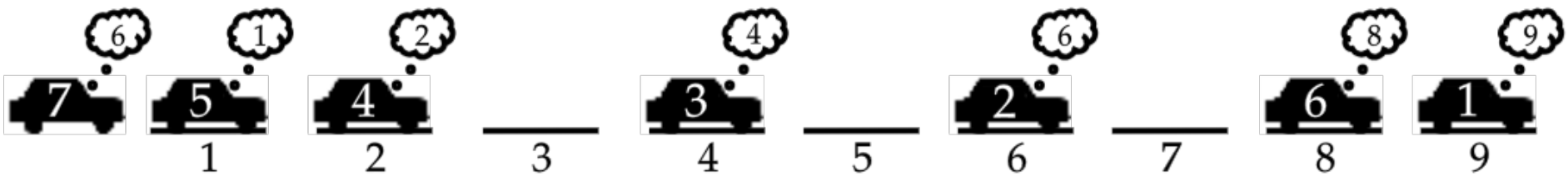
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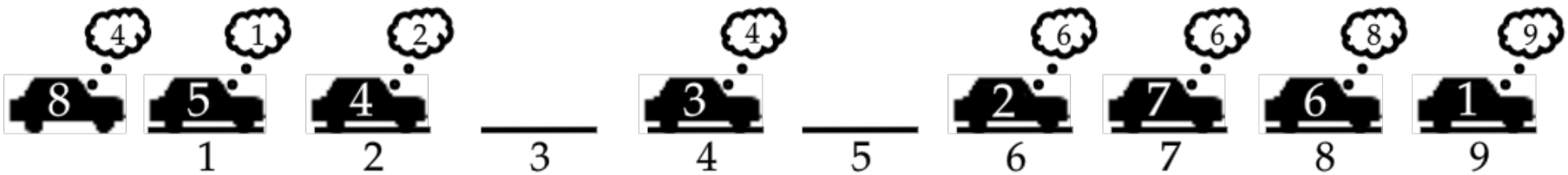
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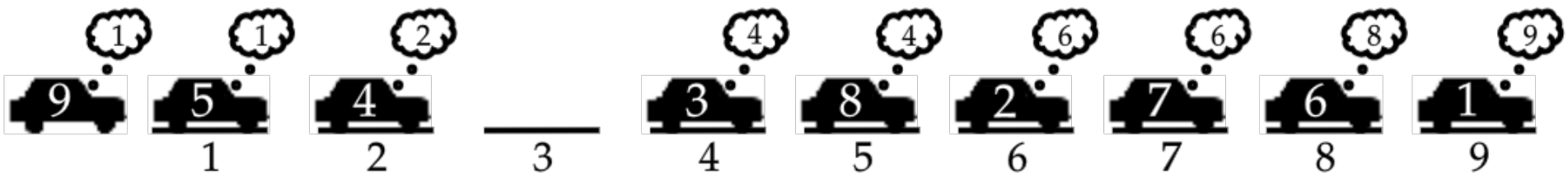
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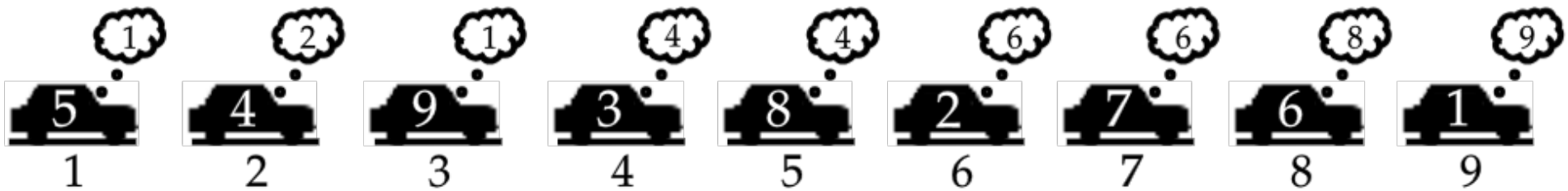
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Is $\alpha = (9, 6, 4, 2, 1, 8, 6, 4, 1)$ a parking function? Yes!



The *parking permutation* of α is

$$\pi = 549382761$$

Number of parking functions

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Theorem (Pyke (1959), Konheim and Weiss (1966))

$|PF(n)| = (n+1)^{(n-1)}$.

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Definition

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- If $\alpha = (9, 6, 4, 2, 1, 8, 6, 4, 1)$ then $\pi = \boxed{5}\boxed{4}9\boxed{3}8\boxed{2}7\boxed{6}\boxed{1}$

First Constraint

Fixed lucky set

Fix a set of lucky cars, can we characterize which parking permutations are possible?

Previous results

Let $\pi \in \mathfrak{S}_n$ with a fixed lucky set I (the cars in the set I are the only lucky cars of π).

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   Example: For $I = \{1, 2, 3, 4, 5, 6\}$,

$$\pi = 5\,|49\,|38\,|27\,|6\,|1$$

Second Constraint

Permutations

A permutation of $[n] = \{1, 2, \ldots, n\}$ is a bijection from $[n]$ to itself.

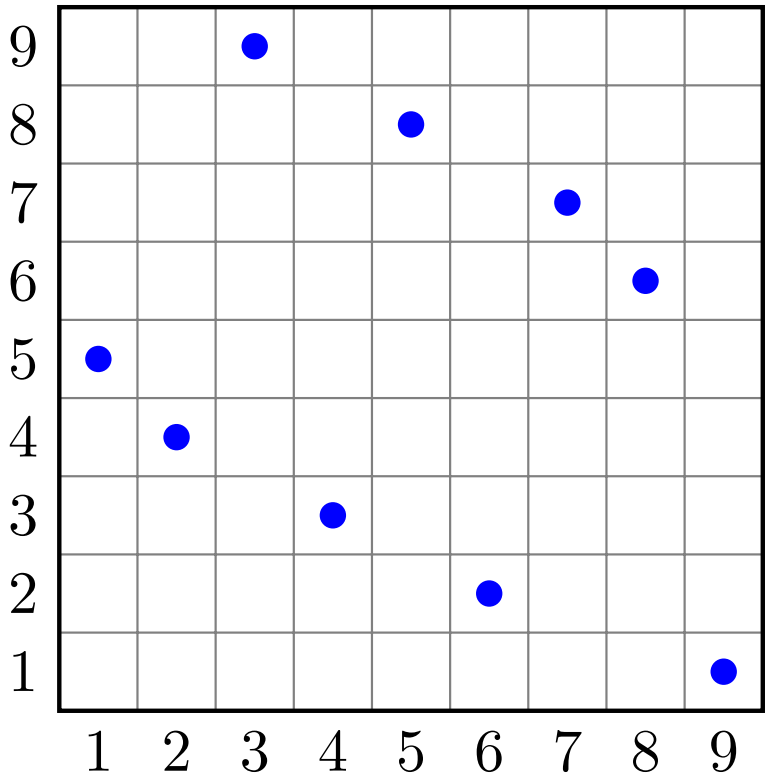


Figure: E.g. $\pi = 549382761$

Patterns in permutations

A permutation σ of size k is a pattern of a permutation π of size n if there exist $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ is in the same relative order ($\equiv$) as σ.

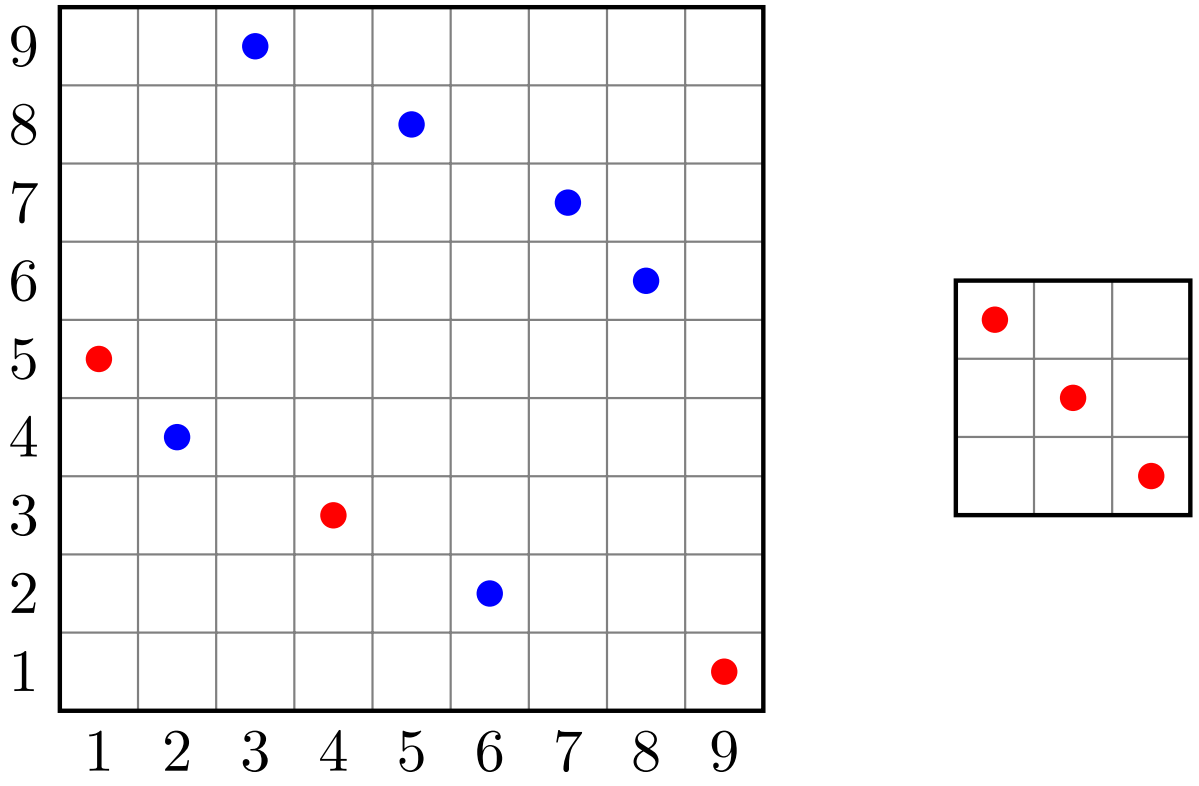


Figure: E.g. 321 is a pattern of $\pi = 549382761$ since $531 \equiv 321$

Pattern Avoidance

A permutation σ of size k is a pattern of a permutation π of size n if there exist $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ is in the same relative order ($\equiv$) as σ.

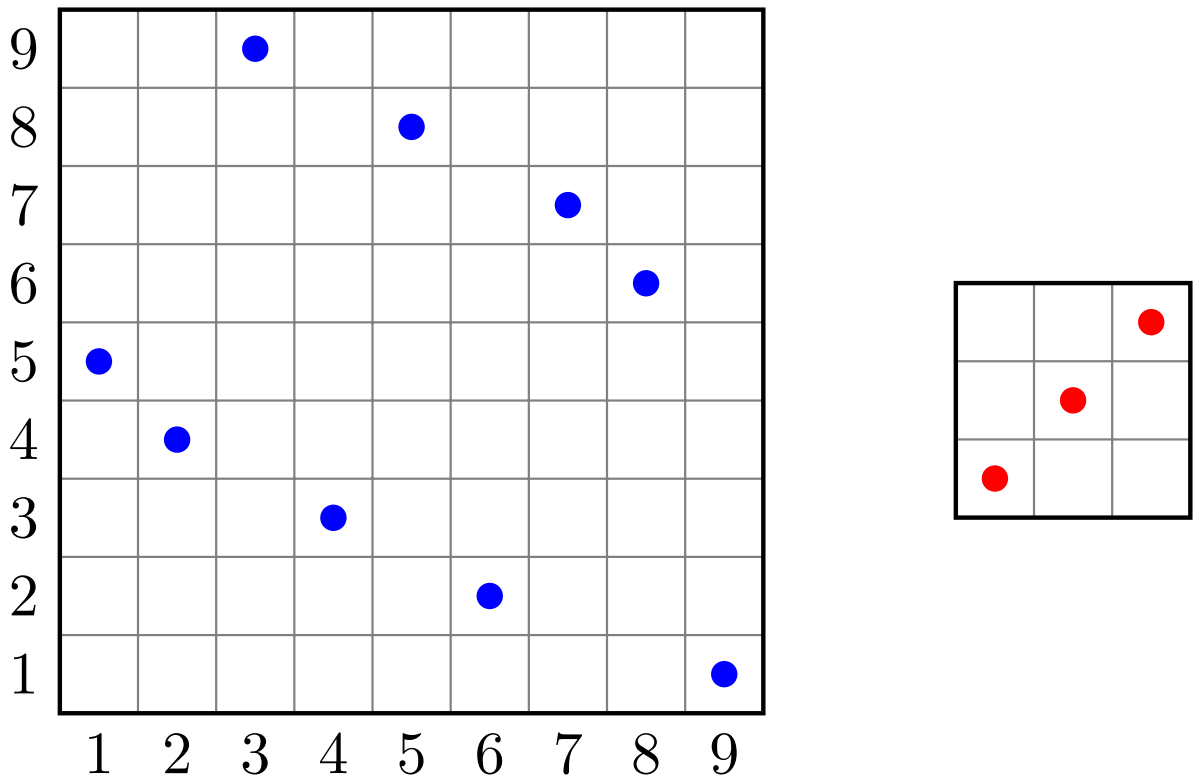


Figure: $\pi = 549382761$ avoids the pattern 123

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Result (Knuth)

For any $\sigma \in \mathfrak{S}_3 = \{123, 132, 213, 231, 312, 321\}$,

$$|\mathsf{Av}_n(\sigma)| = \frac{1}{n+1}\binom{2n}{n} = C_n,$$

where C_n is the nth Catalan number.
The first terms of the sequence are:

(OEIS: A108) $1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots$

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Let $n = 5$. The following are the only parking permutations that avoid the pattern 123 for which the first 3 cars are lucky:

25143 32514 35142 35214 35241

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We focus on the case when $I = \{1, 2, \ldots, k\} = [k]$.

Enumeration of 123-avoiding parking permutations with lucky set $[k]$

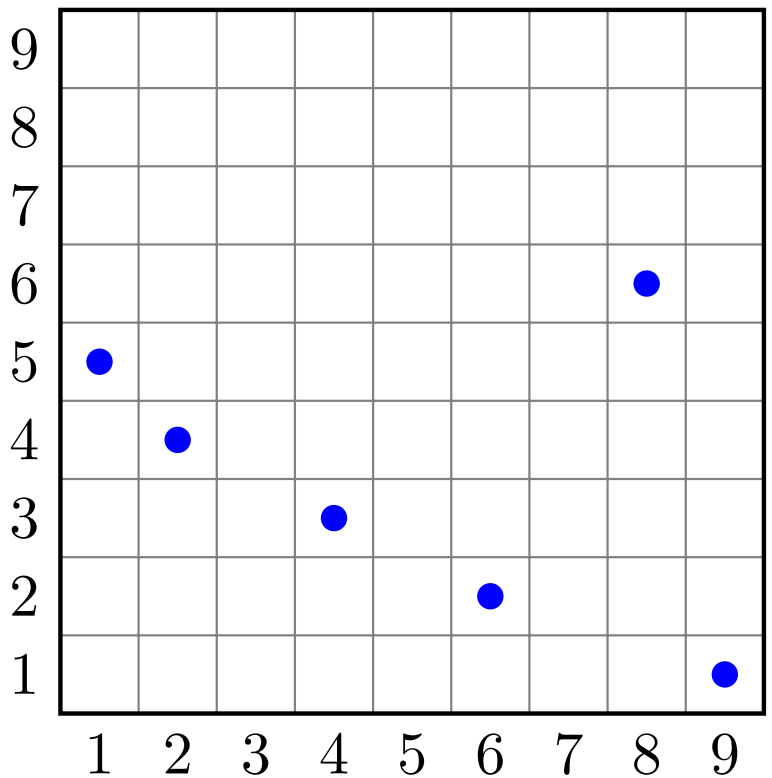
Theorem (Ferreri, M. (2026))

(Lobb numbers - OEIS A039599) The number of parking permutations in $\mathfrak{S}_n$ avoiding the pattern 123 with lucky set $I = \{1, 2, \ldots, k\}$ is given by

$$\frac{2n - 2k + 1}{n + 1} \binom{2k}{n}.$$

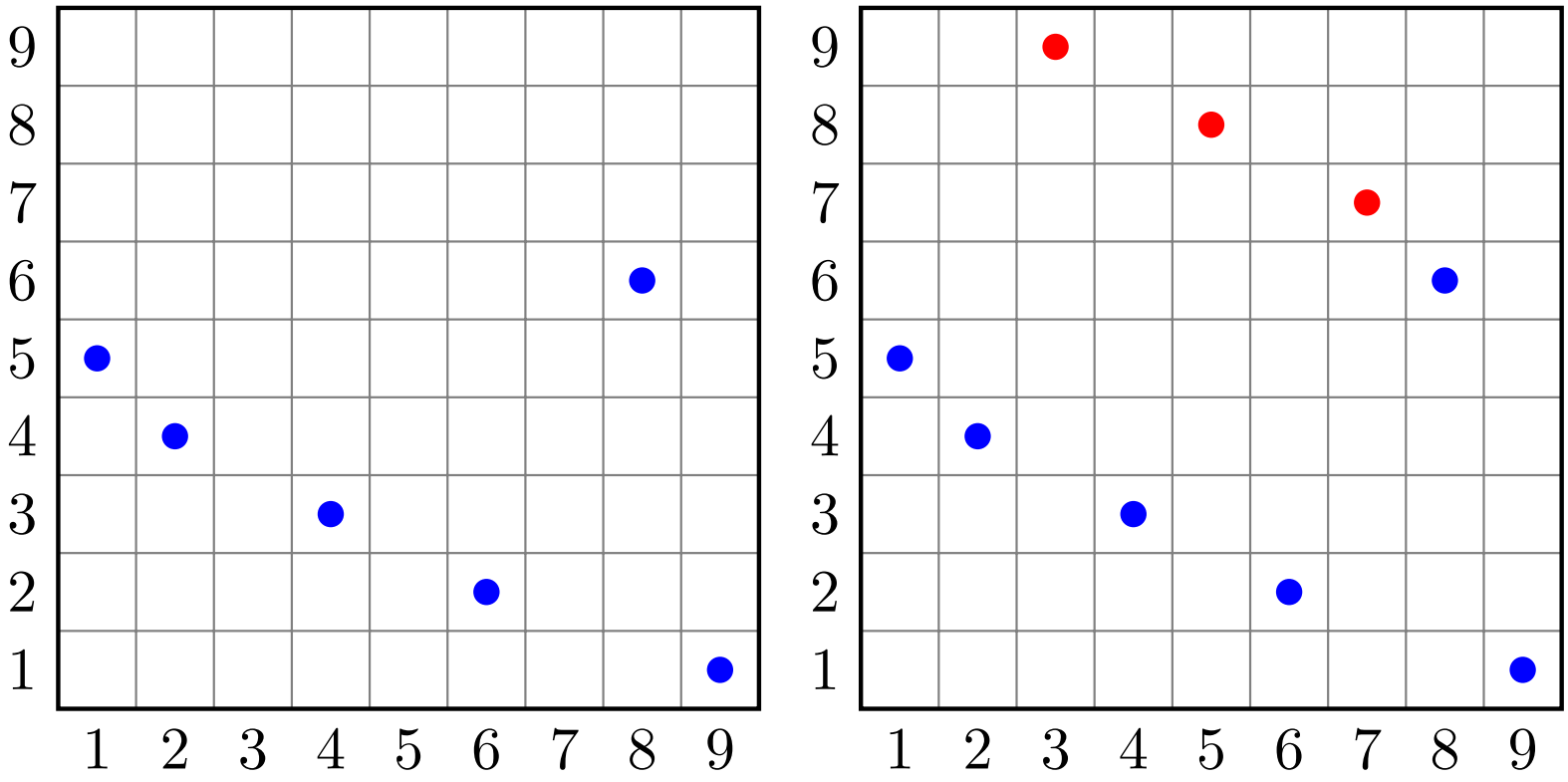
Proof idea

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Structure of 123-avoiding parking permutations with lucky set $[k]$

Start with $\sigma \in \mathsf{Av}_k(123)$:

$$\sigma_1 \quad |\sigma_2 \quad |\sigma_3 \quad \cdots \quad |\sigma_{k-1} \quad |\sigma_k$$

Structure of 123-avoiding parking permutations with lucky set $[k]$

Need to insert the elements $U = \{k+1, k+2, \ldots, n\}$ anywhere after σ_1.

$$\sigma_1 \underbrace{\nearrow} | \sigma_2 \underbrace{\nearrow} \quad | \sigma_3 \underbrace{\nearrow} \quad \cdots \quad | \sigma_{k-1} \underbrace{\nearrow} \quad | \sigma_k \underbrace{\nearrow}$$

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Let τ_i, τ_j be unlucky cars. At most one element can be placed after any lucky car:

$$\sigma_1\tau_i\tau_j|\sigma_2 \quad |\sigma_3\tau_2 \quad \cdots \quad |\sigma_{k-1} \quad |\sigma_k$$

Structure of 123-avoiding parking permutations with lucky set $[k]$

Need to insert the elements $U = \{k+1, k+2, \ldots, n\}$ anywhere after σ_1.

Since all elements in U are greater than any σ_i then whenever $\tau_i < \tau_j$ then the following cannot happen

$$\sigma_1|\sigma_2\tau_i \quad |\sigma_3\tau_2 \quad \cdots \quad |\sigma_{k-1} \quad |\sigma_k\tau_j$$

as $\sigma_1\tau_i\tau_j \equiv 123$.

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Need to insert the elements $U = \{k+1, k+2, \ldots, n\}$ anywhere after σ_1.

Once we choose the positions for the elements in U, their order is fixed as they must appear in decreasing order from left to right. For $\tau_i < \tau_j$,

$$\sigma_1 | \sigma_2 \tau_j \quad | \sigma_3 \tau_2 \quad \cdots \quad | \sigma_{k-1} \quad | \sigma_k \tau_i$$

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Question: Can we we choose any $n-k$ positions?

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If $A(k,i)$ be the number of 123-avoiding permutations (of length k) whose first ascent occurs at position i then

$$\sum_{i=1}^{n-k} \binom{i}{n-k} A(k,i)$$

enumerates the 123-avoiding parking permutations with lucky set $[k]$.

The location of the first ascent in $\mathsf{Av}_n(123)$

Theorem (Connolly, Gabor, and Godbole (2015))

(k-fold Catalan convolution - OEIS A033184) The number of 123-avoiding permutations (of length n) whose first ascent is at position k is equal to the k-fold Catalan convolution:

$$C_{n,k} = \sum_{\substack{i_1+\cdots+i_k=n,\\ i_j\geq 1}} \prod_{r=1}^{k} C_{i_r-1} = \frac{k}{2n-k}\binom{2n-k}{n}.$$

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Hence, $A(k,i) = C_{k,i}$, and

$$\sum_{i=1}^{n-k} \binom{i}{n-k} C_{k,i} = \frac{2n-2k+1}{n+1}\binom{2k}{n}.$$

Thank You!

https://marti310.github.io/research.html