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On the mathematical legacy of Jesús Guillera

RUTGERS EXPERIMENTAL MATHEMATICS SEMINAR

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University of Antwerp

A short pre-Ramanujan history of series for $\frac{1}{\pi}$?

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Theorie der Modular-Functionen und der Modular-Integrale.

(Von Herrn Dr. *Gudermann* zu Münster.)

$$1.\quad K = \tfrac{1}{2}\pi.\left(1 + \frac{1^2}{2^2}.k^2 + \frac{1^2.3^2}{2^2.4^2}.k^4 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}.k^6 + \frac{1^2.3^2.5^2.7^2}{2^2.4^2.6^2.8^2}.k^8 + \text{etc.}\right),$$

$$2.\quad E = \tfrac{1}{2}\pi.\left(1 - \frac{1}{2^2}.k^2 - \frac{1^2.3}{2^2.4^2}.k^4 - \frac{1^2.3^2.5}{2^2.4^2.6^2}.k^6 - \frac{1^2.3^2.5^2.7}{2^2.4^2.6^2.8^2}.k^8 - \text{etc.}\right),$$

Anmerkung. Auch die Convergenz der Reihen (1.) und (2.) wird schwach, wenn sich der Modul k der Grenze Eins nähert. Setzen wir wirklich $k=1$, so ist nach G. T. d. P. F. §. 62. $\frac{2}{\pi}$ die Grenze, welcher sich die numerischen Coefficienten in der Reihe (1.) ohne Ende nähern. Einen höhern Grad der Kleinheit erreichen sie also nicht, und nur so ist es auch erklärlich, dafs die Summe jener Reihe, oder K selbst, unendlich wird, wenn man $k=1$ setzt. Die Convergenz der Reihe (2.) ist etwas rascher, wenn $k=1$ gesetzt wird.

Journal für die reine und angewandte Mathematik, 19 (1839), 45–83

1839

Die einfachsten periodischen Functionen.

(Von dem Herrn Professor Dr. *Schellbach* zu Berlin.)

$$(4.)\quad \frac{2}{\pi} = 1 - \left(\frac{1}{2}\right)^2 - \frac{1}{3}\cdot\left(\frac{1.3}{2.4}\right)^2 - \frac{1}{5}\cdot\left(\frac{1.3.5}{2.4.6}\right)^2 - \frac{1}{7}\cdot\left(\frac{1.3.5.7}{2.4.6.8}\right)^2 - \dots\dots$$

Journal für die reine und angewandte Mathematik, 48 (1854), 207–236

1854

$$\frac{4}{\pi} = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2\cdot4}\right)^2 + \left(\frac{1\cdot3}{2\cdot4\cdot6}\right)^2 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}\right)^2 + \cdots$$

Catalan 1858

Bauer 1875

Forsyth 1883

Von den Coefficienten der Reihen von Kugelfunctionen einer Variablen.

(Von Herrn *G. Bauer* zu München)

$$\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1.3}{2.4}\right)^3 - 13\left(\frac{1.3.5}{2.4.6}\right)^3 + - \cdots$$

Journal für die Reine und Angewandte Mathematik, 56 (1859), 101–121

1859

ON SERIES FOR $\frac{1}{\pi}$ AND $\frac{1}{\pi^2}$.

By J. W. L. GLAISHER.

Series for $\frac{1}{\pi^2}$.

$$\frac{4}{\pi^2}=1.2.3\left(\frac{1}{2}\right)^4-3.4.7\left(\frac{1}{2.4}\right)^4+5.6.11\left(\frac{1.3}{2.4.6}\right)^4+\&c. \qquad (\S\ 32),$$

$$\frac{4}{\pi^2}=\frac{1}{2}-\frac{5}{1.4}\left(\frac{1}{2}\right)^4-\frac{9}{3.6}\left(\frac{1.3}{2.4}\right)^4-\frac{13}{5.8}\left(\frac{1.3.5}{2.4.6}\right)^4-\&c. \qquad (\S\ 33),$$

$$\frac{8}{\pi^2}=1-3\left(\frac{1}{2}\right)^4-7\left(\frac{1}{2.4}\right)^4-11\left(\frac{1.3}{2.4.6}\right)^4-\&c. \qquad (\S\ 34),$$

$$\frac{8}{\pi^2}=-1+3^3\left(\frac{1}{2}\right)^4+7^3\left(\frac{1}{2.4}\right)^4+11^3\left(\frac{1.3}{2.4.6}\right)^4+\&c. \qquad (\S\ 35).$$

Quarterly Journal of Pure and Applied Mathematics, XXXVII (1905-06), 173–198.

1905

Gosperable formulas

Almost all of the formulas given above are ‘Gosperable’, meaning that Maple (or equivalent) immediately finds a closed form for the n-th partial sum of the series. One exception: Bauer’s series.

Note: mostly series with sum $C\pi^k$. (Wallis!)

Gosperable formulas

Asked 1 year, 9 months ago Modified 6 months ago Viewed 656 times

mathoverflow

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Below we give examples of Gosperable formulas

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^6}{(1)_n^6} \frac{1-12n^2+48n^4}{(1-2n)^3} = \frac{8}{\pi^3}, \tag{1}$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^8}{(1)_n^8} \frac{1-16n^2+96n^4-256n^6}{(1-2n)^4} = \frac{16}{\pi^4}, \tag{2}$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^{10}}{(1)_n^{10}} \frac{1-20n^2+160n^4-640n^6+1280n^8}{(1-2n)^5} = \frac{32}{\pi^5}, \tag{3}$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^4}{(1)_n^4} \frac{8n^2+8n+1}{(n+1)^2} = \frac{16}{\pi^2}, \tag{4}$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^6}{(1)_n^6} \frac{(12n^2+10n+1)(12n^2+18n+7)}{(n+1)^4} = \frac{256}{\pi^3}. \tag{5}$$

From Ramanujan to ...

Baruah N.D., Berndt B.C., Chan H.H., 'Ramanujan's Series for $1/\pi$: A Survey', Amer. Math. Monthly, 116 (7) 567–587, 2009.

Important moment:
Ekhad S., Zeilberger D., 'A WZ proof of Ramanujan's formula for Pi', in: Geometry, Analysis and Mechanics, ed. by J.M. Rassias, World Scientific, Singapore, 107–108, 1994.

Doron's algorithm

Purpose: To determine a linear recurrence satisfied by

$$z(n) = \sum_{k=0}^{\infty} F(n,k)$$

$F(n,k)$ is a hypergeometric term.
ZEILBERGER, if you´re lucky, leads to

$$p_0(n)F(n+1,k) + p_1(n)F(n,k) = G(n,k+1) - G(n,k)$$

($p_0(n)$ and $p_1(n)$ polynomials in n/ $G(n,k)$ hypergeometric). We sum over k:

$$p_0(n)z(n+1) + p_1(n)z(n) = f(n). \qquad (1)$$

First-order recurrence!
Note: if $p_0(n)$ and $p_1(n)$ can be factorized into linear factors, things will simplify to:

$$y(n+1) - y(n) = g(n)$$

with general solution $y(n) = y(0) + \sum_{k=0}^{n-1} g(k)$. We call this process *renormalization*.

An example

Note that BC and BZ, people did this by hand:

(1910) KUMMER

$$\sum_{1}^{\infty} \frac{1}{(2n-1)^2(2n+1)^2\ldots(2n+2p-3)^2} = \frac{3p-1}{2(2p-1)\left(1.3\ldots(2p-1)\right)^2} + \frac{2p^3}{2p-1}\sum \frac{1}{(2n-1)^2(2n+1)^2\ldots(2n+2p-1)^2}$$

E. FABRY

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THÉORIE DES SÉRIES

A TERMES CONSTANTS

$$F(p,n) = \frac{\operatorname{pochhammer}\left(\frac{1}{2}, n\right)^2}{2^{2p}\operatorname{pochhammer}\left(\frac{1}{2}, p\right)^2 \operatorname{pochhammer}\left(\frac{1}{2}+p, n\right)^2}$$

⟶ ZEILBERGER ⟶

$$2p^3\, y(p+1) - (2p-1)\, y(p) = -\frac{\left(\frac{3p}{2} - \frac{1}{2}\right)}{2^{2p}\operatorname{pochhammer}\left(\frac{1}{2}, p\right)^2}$$

An example

Note that BC and BZ, people did this by hand: $\sum_{n=0}^{\infty} \frac{1}{2^{2n}}(3n+2)\frac{n!^3}{\left(\frac{3}{2}\right)_n^3} = \frac{\pi^2}{4}$

(1910) KUMMER

$$\sum_{1}^{\infty} \frac{1}{(2n-1)^2(2n+1)^2\ldots(2n+2p-3)^2} = \frac{3p-1}{2(2p-1)\left(1.3\ldots(2p-1)\right)^2} + \frac{2p^3}{2p-1}\sum \frac{1}{(2n-1)^2(2n+1)^2\ldots(2n+2p-1)^2}$$

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THÉORIE DES SÉRIES

A TERMES CONSTANTS

$$F(p, n) = \frac{\mathrm{pochhammer}\left(\frac{1}{2}, n\right)^2}{2^{2p}\,\mathrm{pochhammer}\left(\frac{1}{2}, p\right)^2 \mathrm{pochhammer}\left(\frac{1}{2}+p, n\right)^2}$$

$\longrightarrow$ ZEILBERGER $\longrightarrow$ $2p^3 y(p+1) - (2p-1)\, y(p) = -\frac{\left(\frac{3p}{2}-\frac{1}{2}\right)}{2^{2p}\,\mathrm{pochhammer}\left(\frac{1}{2}, p\right)^2}$

Some nice applications

- $_3F_2(1,1,1;n,n;1) = \frac{(2n-4)!}{6}\left(\frac{n-1}{(n-2)!}\right)^2\left(\pi^2 - 18\sum_{k=0}^{n-3}\frac{k!^2}{(2k+2)!}\right) /; n-2 \in \mathbb{N}$

or in a nicer format: ($(a)_k$ pochhammer notation, $_pF_q$ general. hypergeometric function)

$$\frac{1}{2^{2n}}\frac{(1)_n}{(n+1)^2\left(\frac{1}{2}\right)_n}\,{}_3F_2\left(\begin{matrix}1,1,1\\2+n,2+n\end{matrix};1\right) = \frac{\pi^2}{6} - 3\sum_{k=0}^{n-1}\frac{k!^2}{(2k+2)!}$$

(Start with $F(n,k) = \frac{(1)_k^2}{(2+n)_k^2}$, and apply ZEILBERGER, renormalize and solve the first-order recurrence.)

Note that taking $n = 0$ solves the Basel problem!

Some nice applications

Other ones not found on the Wolfram function site:

$$2(-1)^n \frac{\left(\frac{1}{2}\right)_n^3}{(n-1)!n!^2} {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}+n, 1\\ 1+n, 1+n\end{matrix}; 1\right) = \frac{2}{\pi} - \sum_{k=0}^{n-1} (-1)^k (4k+1) \frac{\left(\frac{1}{2}\right)_k^3}{k!^3}$$

$$2^3 \left(\frac{1}{2^2}\right)^n \frac{\left(\frac{1}{2}\right)_n^3}{(n-1)!n!^2} {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, 1\\ 1+n, 1+n\end{matrix}; 1\right) = \frac{4}{\pi} - \sum_{k=0}^{n-1} (6k+1) \frac{\left(\frac{1}{2}\right)_k^3}{4^k k!^3}$$

$$2^4 \left(-\frac{1}{2^2}\right)^n \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n}{(n-1)!n!^2} {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}+n, 1\\ 1+n, 1+2n\end{matrix}; 1\right) = \frac{8}{\pi} - \sum_{k=0}^{n-1} (-1)^k (20k+3) \frac{\left(\frac{1}{4}\right)_k \left(\frac{1}{2}\right)_k \left(\frac{3}{4}\right)_k}{4^k k!^3}$$

$$2^5 \left(\frac{1}{2^6}\right)^n \frac{\left(\frac{1}{2}\right)_n^3}{(n-1)!n!^2} {}_3F_2\left(\begin{matrix}\frac{1}{2}+n, \frac{1}{2}+n, 1\\ 1+2n, 1+2n\end{matrix}; 1\right) = \frac{16}{\pi} - \sum_{k=0}^{n-1} (42k+5) \frac{\left(\frac{1}{2}\right)_k^3}{64^k k!^3}$$

Some nice applications

Other ones not found on the Wolfram function site:

$$4\frac{\left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n^3}{(n-1)!^3 n!}\,{}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{2}-n,1+n\end{matrix};1\right)=-\frac{1}{\pi}\sum_{k=0}^{n-1}(-1)^k(4k+1)\frac{\left(\frac{1}{2}\right)_k^3}{k!^3}$$

$$16\frac{\left(-\frac{1}{4}\right)_n\left(\frac{1}{4}\right)_n\left(\frac{1}{2}\right)_n}{(n-1)!^3}\,{}_3F_2\left(\begin{matrix}\frac{1}{2}-n,\frac{1}{2}-n,\frac{1}{2}\\ \frac{3}{2}-2n,1\end{matrix};1\right)=-\frac{1}{\pi}\sum_{k=0}^{n-1}(6k+1)\frac{\left(\frac{1}{2}\right)_k^3}{4^k k!^3}$$

$$64\frac{\left(-\frac{1}{4}\right)_n\left(\frac{1}{4}\right)_n^2\left(\frac{3}{4}\right)_n}{(n-1)!^3 n!}\,{}_3F_2\left(\begin{matrix}\frac{1}{2}-n,\frac{1}{2},\frac{1}{2}\\ \frac{3}{2}-2n,1+n\end{matrix};1\right)=-\frac{1}{\pi}\sum_{k=0}^{n-1}(-1)^k(20k+3)\frac{\left(\frac{1}{4}\right)_k\left(\frac{1}{2}\right)_k\left(\frac{3}{4}\right)_k}{4^k k!^3}$$

$$256\frac{\left(\frac{27}{64}\right)^n\left(-\frac{1}{6}\right)_n\left(\frac{1}{6}\right)_n\left(\frac{1}{2}\right)_n}{(n-1)!^3}\,{}_3F_2\left(\begin{matrix}\frac{1}{2}-n,\frac{1}{2}-n,\frac{1}{2}-n\\ \frac{3}{2}-3n,1\end{matrix};1\right)=-\frac{1}{\pi}\sum_{k=0}^{n-1}(42k+5)\frac{\left(\frac{1}{2}\right)_k^3}{64^k k!^3}$$

Series with sum $C\frac{1}{\pi^2}$

Remember: Glaisher's series for $\frac{1}{\pi^2}$ are all Gosperable.

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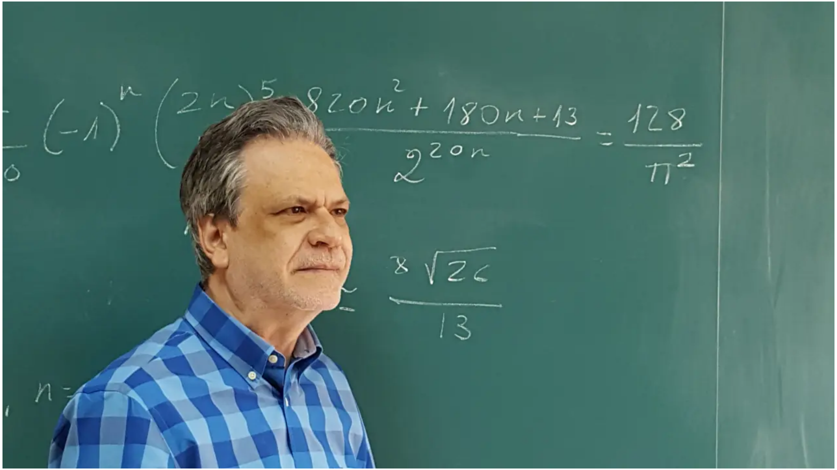
Jesús Guillera, in 2016



1955-2026

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Jesús Guillera, in 2015

EL PAÍS

ningún patrón a lo largo de sus infinitas cifras. "Guillera es nuestro Ramanujan español", sentencia Javier Cilleruelo, miembro del Instituto de Ciencias Matemáticas (ICMAT), en Madrid.

Aquel día de 2002, Guillera se armó de valor para enviar por correo electrónico sus primeras fórmulas al matemático estadounidense Doron Zeilberger, referencia internacional en este campo. A la media hora, el investigador contestó con entusiastas felicitaciones desde su despacho de la Universidad de Rutgers.

"Guillera is our Spanish Ramanujan," declares Javier Cilleruelo, a member of the Institute of Mathematical Sciences (ICMAT) in Madrid.
On that day in 2002, Guillera plucked up the courage to email his first formulas to the American mathematician Doron Zeilberger, an international authority in the field. Within half an hour, the researcher replied with enthusiastic congratulations from his office at Rutgers University.

Jesús Guillera's email to Doron Zeilberger, in 2002

From jguillera@able.es Thu Mar 7 14:51:40 2002
To: <zeilberg@math.rutgers.edu>
Subject: formulas
Date: Thu, 7 Mar 2002 20:45:25 +0100

Dear Doron,

I have found the following formulae and others using the WZ method.

$$\begin{cases} \dfrac{8}{\pi^2} = \displaystyle\sum_{n=0}^{\infty} (-1)^n (20n^2+8n+1) \frac{\left(\frac{1}{2}\right)_n^5}{2^{2n} n!^5}, \\ \dfrac{32}{\pi^2} = \displaystyle\sum_{n=0}^{\infty} (120n^2+34n+3) \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n^3 \left(\frac{3}{4}\right)_n}{2^{4n} n!^5}, \\ \dfrac{128}{\pi^2} = \displaystyle\sum_{n=0}^{\infty} (-1)^n (820n^2+180n+13) \frac{\left(\frac{1}{2}\right)_n^5}{2^{10n} n!^5}. \end{cases}$$

I am surprised because I have not seen in internet formulas like this with pi square in the denominator.
What is your opinion?

I have also found all the formulas of Ramanujan with C(2n,n) power 3 using also the WZ method.

With friendly greetings,

Jesus Guillera

Series with sum $C\frac{1}{\pi^2}$

These 3 series are the first non-trivial ($=$ non-Gosperable) series for $\frac{1}{\pi^2}$ ever, I think.

Jesús Guillera, 'About a New Kind of Ramanujan-Type Series', Experiment. Math. 12 (4) 507–510, 2003.

Series with sum $C\frac{1}{\pi^2}$

In a paper published in 2008 Guillera gave formulas relating series of Ramanujan to these series for $\frac{1}{\pi^2}$. Using ZEILBERGER we can make this relation very clear!

Let $f(a)$ and $g(a)$ be the functions:

$$f(a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}} \frac{(a+\frac{1}{2})_n^5}{(a+1)_n^5}[820(n+a)^2 + 180(n+a) + 13],$$

$$g(a) = \sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{(a+\frac{1}{2})_n^3}{(a+1)_n^3}[42(n+a) + 5],$$

then we have

$$f(a) = \frac{8}{\pi} \cdot \frac{16^a}{\cos \pi a} \cdot \frac{(1)_a^2}{(\frac{1}{2})_a^2} \cdot g(a) + \frac{2048a^3}{2a-1} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}+a)_n^3}{(2a+1)_n^2(\frac{3}{2}-a)_n}$$

Jesús Guillera, ‘Hypergeometric identities for 10 extended Ramanujan-type series’, Ramanujan J. 15 (2) 219–234, 2008.

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Series with sum $C\frac{1}{\pi^2}$

$$2^5\left(-\frac{1}{2^2}\right)^k\frac{\left(-\frac12\right)_k\left(\frac12\right)_k^4}{(k-1)!^3k!^2}\,{}_4F_3\left(\begin{matrix}\frac12,\frac12,\frac12+k,1\\ \frac32-k,1+k,1+k\end{matrix};1\right)=-\frac{2}{\pi}\sum_{n=0}^{k-1}(6n+1)\frac{\left(\frac12\right)_n^3}{4^n n!^3}+\sum_{n=0}^{k-1}(-1)^n(20n^2+8n+1)\frac{\left(\frac12\right)_n^5}{4^n n!^5}$$

has limit 0 — sums to $\frac{4}{\pi}$ — sums to $\frac{8}{\pi^2}$

Ramanujan + Guillera

$$2^8\left(\frac{1}{2^4}\right)^k\frac{\left(-\frac12\right)_k\left(\frac14\right)_k\left(\frac12\right)_k^2\left(\frac34\right)_k}{(k-1)!^3k!^2}\,{}_4F_3\left(\begin{matrix}\frac12,\frac12+k,\frac12+k,1\\ \frac32-k,1+2k,1+k\end{matrix};1\right)=-\frac{4}{\pi}\sum_{n=0}^{k-1}(-1)^n(20n+3)\frac{\left(\frac14\right)_n\left(\frac12\right)_n\left(\frac34\right)_n}{4^n n!^3}+\sum_{n=0}^{k-1}(120n^2+34n+3)\frac{\left(\frac14\right)_n\left(\frac12\right)_n^3\left(\frac34\right)_n}{16^n n!^5}$$

has limit 0 — sums to $\frac{8}{\pi}$ — sums to $\frac{32}{\pi^2}$

Ramanujan + Guillera

$$2^{11}\left(-\frac{1}{2^{10}}\right)^k\frac{\left(-\frac12\right)_k\left(\frac12\right)_k^4}{(k-1)!^3k!^2}\,{}_4F_3\left(\begin{matrix}\frac12+k,\frac12+k,\frac12+k,1\\ \frac32-k,1+2k,1+2k\end{matrix};1\right)=-\frac{8}{\pi}\sum_{n=0}^{k-1}(42n+5)\frac{\left(\frac12\right)_n^3}{64^n n!^3}+\sum_{n=0}^{k-1}(-1)^n(820n^2+180n+13)\frac{\left(\frac12\right)_n^5}{1024^n n!^5}$$

has limit 0 — sums to $\frac{16}{\pi}$ — sums to $\frac{128}{\pi^2}$

Ramanujan + Guillera

Guillera's series with sum $C\frac{1}{\pi^2}$

There's more!

Starting from some series I published in 2010 (which I would now classify as 'trivial'), we can get the Guillera series by accelerating them.

The series: $$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^4}{(n!)^4} \cdot \frac{4n+1}{(2n-(2k-1)) \cdot \cdots \cdot (2n-1) \cdot (n+1) \cdot \cdots \cdot (n+k)}$$

The accelerated series $(k \to k+n \text{ in } F(n,k))$

$$4\left(-\frac{1}{2^2}\right)^n \frac{\left(\frac{1}{2}\right)_n^5 \left(\frac{5}{4}\right)_n}{(n-1)!\,n!^4 \left(\frac{1}{4}\right)_n} {}_6F_5\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}+n, \frac{1}{2}+n, \frac{1}{2}+n, \frac{5}{4}+n, 1 \\ 1+n, 1+n, 1+n, 1+2n, \frac{1}{4}+n\end{matrix}; 1\right)$$
$$= \frac{8}{\pi^2} - \sum_{k=0}^{n-1} (-1)^k (20k^2+8k+1) \frac{\left(\frac{1}{2}\right)_k^5}{4^k k!^5}$$

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Guillera's series with sum $C\frac{1}{\pi^2}$

There's more!
Starting from some series I published in 2010 (which I would now classify as 'trivial'), we can get the Guillera series by accelerating them.

The series: $$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^4}{(n!)^4} \cdot \frac{4n+1}{(2n-(2k-1))^2 \cdot \cdots \cdot (2n-1)^2 \cdot (n+1)^2 \cdot \cdots \cdot (n+k)^2}$$

The accelerated series

$$32\left(\frac{1}{2^4}\right)^n \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n^3 \left(\frac{3}{4}\right)_n}{(n-1)!n!^4} {}_6F_5\left({\frac{1}{2}, \frac{1}{2}, \frac{1}{2}+n, \frac{1}{2}+n, \frac{5}{4}+n, 1 \atop 1+n, 1+n, 1+2n, 1+2n, \frac{1}{4}+n}; 1 \right)$$
$$= \frac{32}{\pi^2} - \sum_{k=0}^{n-1} (120k^2+34k+3) \frac{\left(\frac{1}{4}\right)_k \left(\frac{1}{2}\right)_k^3 \left(\frac{3}{4}\right)_k}{16^k k!^5}$$

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Guillera's series with sum $C\frac{1}{\pi^2}$

The third one gave some problems, but after a bit of experimenting I found these:

$$8\frac{\left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n^6\left(\frac{3}{2}\right)_n}{(n-1)!^4 n!^4}\,{}_7F_6\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}+n,2n,\frac{5}{4}+\frac{n}{2}\\ 1+n,1+n,1+n,1+n,\frac{3}{2}-n,\frac{1}{4}+\frac{n}{2}\end{matrix};1\right)$$

$$=-\frac{1}{\pi^2}\sum_{k=0}^{n-1}(-1)^k(20k^2+8k+1)\frac{\left(\frac{1}{2}\right)_k^5}{4^k k!^5}$$

$$2^{11}(\text{-}\tfrac{27}{256})^n(2n-1)\frac{\left(-\frac{1}{6}\right)_n\left(\frac{1}{6}\right)_n\left(\frac{1}{2}\right)_n^2}{(n-1)!^4}\,{}_7F_6\left(\begin{matrix}\frac{1}{2}-n,\frac{1}{2}-n,\frac{1}{2}-n,\frac{1}{2}-n,\frac{1}{2}-n,2n,\frac{5}{4}-\frac{n}{2}\\ 1,1,1,1,\frac{3}{2}-3n,\frac{1}{4}-\frac{n}{2}\end{matrix};1\right)$$

$$=\frac{1}{\pi^2}\sum_{k=0}^{n-1}(820k^2+180k+13)\frac{\left(\frac{1}{2}\right)_k^5}{1024^k k!^5}$$

Note that these were found by replacing $F(n,k)$ by $F(n,k-n)$ in the ones we'll see later on.

Guillera's series with sum $C\frac{1}{\pi^2}$

There's more!
Jesús Guillera found another series in 2010.

$$\sum_{n=0}^{\infty}(74n^2+27n+3)\frac{\left(\frac{1}{3}\right)_n\left(\frac{1}{2}\right)_n^3\left(\frac{2}{3}\right)_n}{\left(\frac{64}{27}\right)^n n!^5}=\frac{48}{\pi^2}$$

And again it is related to some of Ramanujan's series!

Jesús Guillera, 'A new Ramanujan-like series for $\frac{1}{\pi^2}$', arXiv, 2010.

Guillera's series with sum $C\frac{1}{\pi^2}$

$$32\left(\frac{27}{64}\right)^k \frac{\left(\frac13\right)_k\left(\frac12\right)_k^3\left(\frac23\right)_k}{(k-1)!^2k!^3}\,{}_4F_3\left(\begin{matrix}\frac12+k,\frac12+k,3k,1\\1+k,1+2k,1+2k\end{matrix};1\right)=\frac{12}{\pi}\sum_{n=0}^{k-1}(6n+1)\frac{\left(\frac12\right)_n^3}{4^n n!^3}-\sum_{n=0}^{k-1}(74n^2+27n+3)\frac{\left(\frac13\right)_n\left(\frac12\right)_n^3\left(\frac23\right)_n}{\left(\frac{64}{27}\right)^n n!^5}$$

has limit 0 (the ${}_4F_3$ term) — sums to $\frac{4}{\pi}$ (Ramanujan) + sums to $\frac{48}{\pi^2}$ (Guillera)

$$128\left(\frac{27}{64}\right)^k \frac{\left(\frac13\right)_k\left(\frac12\right)_k^3\left(\frac23\right)_k}{(k-1)!^2k!^3}\,{}_4F_3\left(\begin{matrix}\frac12,\frac12,3k,1\\1+k,1+k,1+k\end{matrix};1\right)=\frac{3}{\pi}\sum_{n=0}^{k-1}(42n+5)\frac{\left(\frac12\right)_n^3}{64^n n!^3}-\sum_{n=0}^{k-1}(74n^2+27n+3)\frac{\left(\frac13\right)_n\left(\frac12\right)_n^3\left(\frac23\right)_n}{\left(\frac{64}{27}\right)^n n!^5}$$

has limit 0 (the ${}_4F_3$ term) — sums to $\frac{16}{\pi}$ (Ramanujan) + sums to $\frac{48}{\pi^2}$ (Guillera)

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Guillera's series with sum $C\frac{1}{\pi^2}$

$$32\left(\frac{27}{64}\right)^k \frac{\left(\frac13\right)_k\left(\frac12\right)_k^3\left(\frac23\right)_k}{(k-1)!^2k!^3}\,{}_4F_3\left(\begin{matrix}\frac12+k,\frac12+k,3k,1\\1+k,1+2k,1+2k\end{matrix};1\right)=\frac{12}{\pi}\sum_{n=0}^{k-1}(6n+1)\frac{\left(\frac12\right)_n^3}{4^n n!^3}-\sum_{n=0}^{k-1}(74n^2+27n+3)\frac{\left(\frac13\right)_n\left(\frac12\right)_n^3\left(\frac23\right)_n}{\left(\frac{64}{27}\right)^n n!^5}$$

has limit 0; sums to $\frac{4}{\pi}$; sums to $\frac{48}{\pi^2}$

Ramanujan + Guillera

$$128\left(\frac{27}{64}\right)^k \frac{\left(\frac13\right)_k\left(\frac12\right)_k^3\left(\frac23\right)_k}{(k-1)!^2k!^3}\,{}_4F_3\left(\begin{matrix}\frac12,\frac12,3k,1\\1+k,1+k,1+k\end{matrix};1\right)=\frac{3}{\pi}\sum_{n=0}^{k-1}(42n+5)\frac{\left(\frac12\right)_n^3}{64^n n!^3}-\sum_{n=0}^{k-1}(74n^2+27n+3)\frac{\left(\frac13\right)_n\left(\frac12\right)_n^3\left(\frac23\right)_n}{\left(\frac{64}{27}\right)^n n!^5}$$

has limit 0; sums to $\frac{16}{\pi}$; sums to $\frac{48}{\pi^2}$

Ramanujan + Guillera

$$32\frac{\left(\frac14\right)_k^2\left(\frac34\right)_k^2}{(k-1)!k!^3}\,{}_4F_3\left(\begin{matrix}\frac12+k,\frac12+2k,3k,1\\1+k,1+2k,1+3k\end{matrix};1\right)=\frac{24}{\pi}\sum_{n=0}^{k-1}\frac{\left(\frac14\right)_n\left(\frac34\right)_n}{(2n+1)n!^2}-\sum_{n=0}^{k-1}(288n^3+320n^2+103n+9)\frac{\left(\frac14\right)_n^2\left(\frac34\right)_n^2}{n!^4(1+2n)(1+3n)(2+3n)}$$

sums to $\frac{4\ln(1+\sqrt2)}{\pi}$; sums to $\frac{48\ln(3)}{\pi^2}$ (?)

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The series with sum $C\frac{1}{\pi^4}$

Until recently the two known series with sum $\frac{1}{\pi^4}$ remained unproven:

$$\sum_{k=0}^{\infty}(43680k^4+20632k^3+4340k^2+466k+21)\frac{\left(\frac{1}{4}\right)_k\left(\frac{1}{2}\right)_k^7\left(\frac{3}{4}\right)_k}{4096^k k!^9}=\frac{2048}{\pi^4}$$

$$\sum_{k=0}^{\infty}(4528k^4+3180k^3+972k^2+147k+9)\frac{\left(\frac{1}{4}\right)_k\left(\frac{1}{3}\right)_k\left(\frac{1}{2}\right)_k^5\left(\frac{2}{3}\right)_k\left(\frac{3}{4}\right)_k}{\left(-\frac{256}{27}\right)^k k!^9}=\frac{768}{\pi^4}$$

Enters Kam Cheong Au. (Thank you, Kam Cheong Au!)

Guillera's series and the series with sum $C\frac{1}{\pi^4}$

Using ZEILBERGER we can easily prove the following:

$$\frac{2^{14}}{2^{12n}}\frac{(6n+1)\left(\frac{1}{4}\right)_n\left(\frac{1}{2}\right)_n^7\left(\frac{3}{4}\right)_n}{(2n-1)(n-1)!^4 n!^5}\,{}_8F_7\left(\begin{matrix}\frac{1}{2}+n,\frac{1}{2}+n,\frac{1}{2}+n,\frac{1}{2}+n,\frac{1}{2}+n,\frac{5}{4}+\frac{3}{2}n,4n,1\\ \frac{3}{2}-n,1+2n,1+2n,1+2n,1+2n,1+2n,\frac{1}{4}+\frac{3}{2}n\end{matrix};1\right)$$

$$=\frac{16}{\pi^2}\sum_{k=0}^{n-1}(-1)^k(820k^2+180k+13)\frac{\left(\frac{1}{2}\right)_k^5}{1024^k k!^5}$$

$$-\sum_{k=0}^{n-1}(43680k^4+20632k^3+4340k^2+466k+21)\frac{\left(\frac{1}{4}\right)_k\left(\frac{1}{2}\right)_k^7\left(\frac{3}{4}\right)_k}{4096^k k!^9}$$

Note that we can prove that $\lim_{n\to\infty} LHS = 0$ leading to a proof of Jim Cullen's formula, thanks to Jesús Guillera's series.

Guillera's series and the series with sum $C\frac{1}{\pi^4}$

Using ZEILBERGER we can also prove that:

$$2^9(\text{-}\tfrac{3^3}{2^8})^n \frac{(6n+1)\left(\frac{1}{4}\right)_n\left(\frac{1}{3}\right)_n\left(\frac{1}{2}\right)_n^5\left(\frac{2}{3}\right)_n\left(\frac{3}{4}\right)_n}{(n-1)!^4 n!^5} {}_8F_7\left(\begin{matrix}\frac{1}{2}+n,\frac{1}{2}+n,\frac{1}{2}+n,\frac{1}{2}+n,\frac{1}{2}+2n,\frac{5}{4}+\frac{3}{2}n,3n,1\\ \frac{3}{2},1+n,1+2n,1+2n,1+2n,1+2n,\frac{1}{4}+\frac{3}{2}n\end{matrix};1\right)$$

$$=\frac{96}{\pi^2}\sum_{k=0}^{n-1}(-1)^k(20k^2+8k+1)\frac{\left(\frac{1}{2}\right)_k^5}{4^k k!^5}$$

$$-\sum_{k=0}^{n-1}(4528k^4+3180k^3+972k^2+147k+9)\frac{\left(\frac{1}{4}\right)_k\left(\frac{1}{3}\right)_k\left(\frac{1}{2}\right)_k^5\left(\frac{2}{3}\right)_k\left(\frac{3}{4}\right)_k}{\left(-\frac{256}{27}\right)^k k!^9}$$

This leads to a proof of Yue Zhao's formula, by way of Jesús Guillera's series.

A new series?

By experimenting with the arguments of the hypergeometric function, we found this:

$$8(6n+1)\frac{\left(\frac{1}{4}\right)_n^2\left(\frac{1}{2}\right)_n^3\left(\frac{3}{4}\right)_n^2}{(n-1)!n!^5\left(\frac{3}{2}\right)_n}\,{}_8F_7\left(\begin{matrix}\frac{1}{2}+n,\frac{1}{2}+n,\frac{1}{2}+n,\frac{1}{2}+2n,\frac{1}{2}+2n,\frac{5}{4}+\frac{3}{2}n,2n,1\\ \frac{3}{2}+n,1+n,1+n,1+2n,1+2n,1+2n,\frac{1}{4}+\frac{3}{2}n\end{matrix};1\right)$$
$$=\frac{32}{\pi^2}\sum_{k=0}^{n-1}\frac{\left(\frac{1}{2}\right)_k^2}{(2k+1)k!^2}-\sum_{k=0}^{n-1}(96k^2+32k+3)\frac{\left(\frac{1}{4}\right)_k^2\left(\frac{1}{2}\right)_k^3\left(\frac{3}{4}\right)_k^2}{k!^6\left(\frac{3}{2}\right)_k}$$

Using the fact that $\lim_{n\to\infty} LHS = 0$ (thanks, John) and Entry 29(a) in Ramanujan's second notebook :

$$\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2}{(2k+1)k!^2}=\frac{4G}{\pi}$$

we get:

$$\sum_{k=0}^{\infty}(96k^2+32k+3)\frac{\left(\frac{1}{4}\right)_k^2\left(\frac{1}{2}\right)_k^3\left(\frac{3}{4}\right)_k^2}{k!^6\left(\frac{3}{2}\right)_k}=\frac{128G}{\pi^3}.$$

Jesús Guillera, 1 July 2024

Questions:

1-) We ask for other examples of Gospereable formulas not very similar to the given above.

2-) Are all hypergeometric series with $z = 1$ and a known sum Gospereable?

3-) Is it possible for some of the examples to introduce a free parameter k such that they turn WZ (Wilf-Zeilberger) provable? The advantage of it will be that the generalization may allow improving the rate of convergence by using WZ-transformations, for example $k \to k + n$.

Thank you!

Addendum

While experimenting with upside-down series, I found this:

$$\left(-\tfrac{27}{256}\right)^n(3n+2)\frac{n!^5\left(\frac{3}{4}\right)_n\left(\frac{5}{6}\right)_n\left(\frac{7}{6}\right)_n\left(\frac{5}{4}\right)_n}{\left(\frac{1}{2}\right)_n^4\left(\frac{3}{2}\right)_n^5}\,{}_8F_7\left(\begin{matrix}1+n,1+n,1+n,1+n,\frac{3}{2}+2n,\frac{3}{2}+3n,2+\frac{3}{2}n,1\\ \frac{3}{2},\frac{3}{2}+n,2+2n,2+2n,2+2n,2+2n,1+\frac{3}{2}n\end{matrix};1\right)$$

$$=2\zeta(3)-4\sum_{k=0}^{n-1}(-256)^k(10k^2+14k+5)\frac{1}{(2k+1)^5\binom{2k}{k}^5}$$

$$+\frac{1}{32}\sum_{k=0}^{n-1}(2264k^4+6118k^3+6267k^2+2884k+503)\frac{k!^5\left(\frac{3}{4}\right)_k\left(\frac{5}{6}\right)_k\left(\frac{7}{6}\right)_k\left(\frac{5}{4}\right)_k}{\left(-\frac{256}{27}\right)^k\left(\frac{3}{2}\right)_k^9}$$

The second term at the RHS sums to $-14\zeta(3)$ and the third one to $12\zeta(3)$!