Counting Colored Trees

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Rutgers Experimental Mathematics Seminar
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Joint work with Stoyan Dimitrov, Kimberly Hadaway, Ashley Tharp, and Stephan Wagner

Acknowledgment

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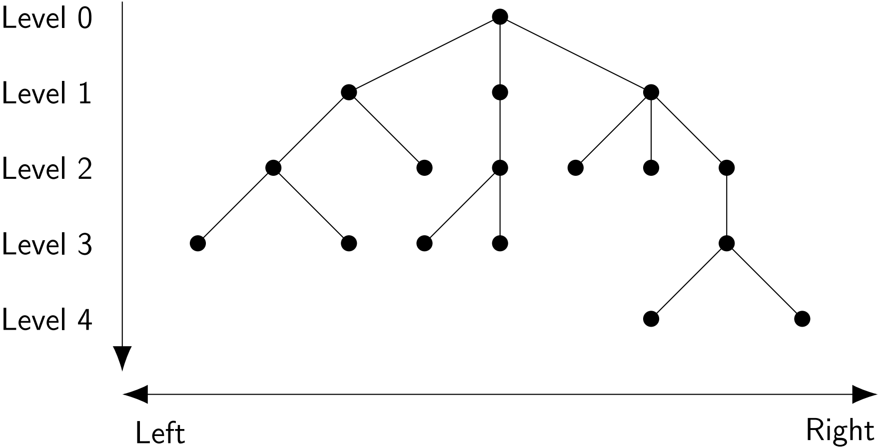
Plane Trees



Classical result

Number of plane trees with n vertices $= C_{n-1}$ (Catalan numbers)

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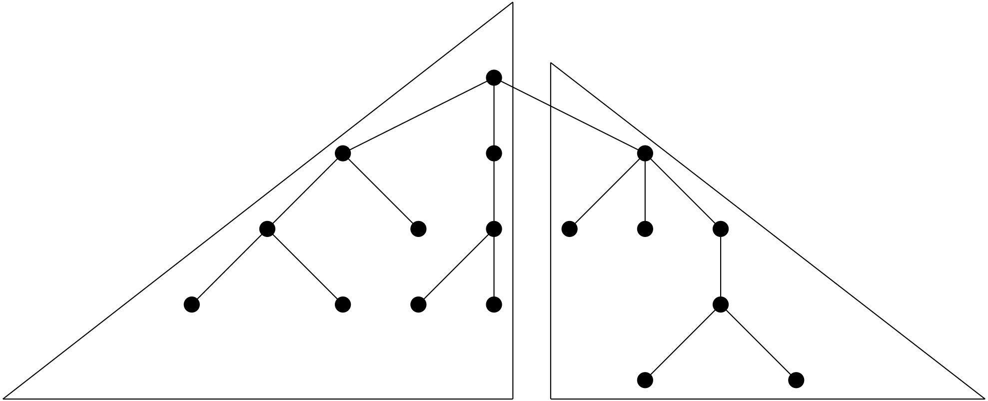
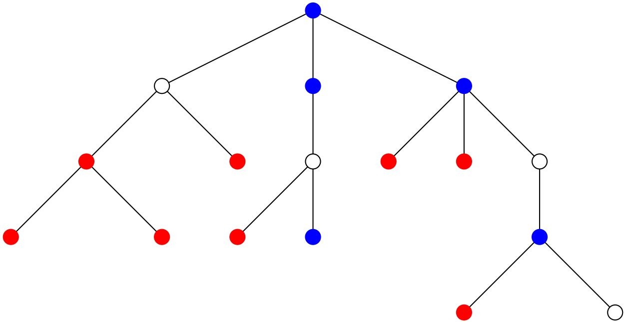
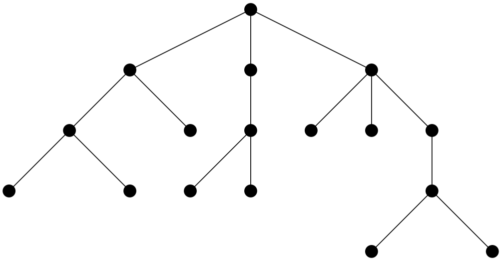


Colored Plane Trees



Rules for coloring plane trees

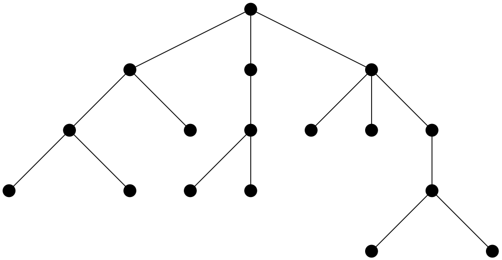
How to color vertices of a plane tree:



Rules for coloring plane trees

How to color vertices of a plane tree:

- How many/which colors to use?



1. Blue
2. Red
3. White

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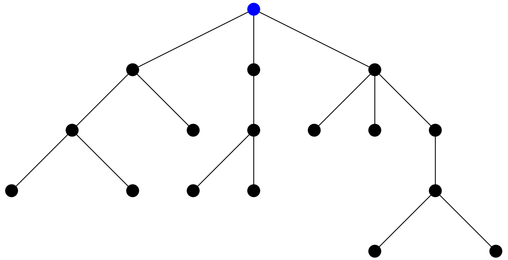

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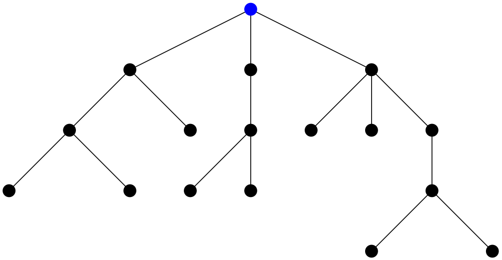

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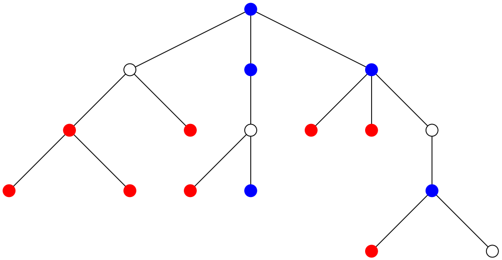
Example:

- Blue followed by any color
- Red only followed by red
- White followed by red or blue



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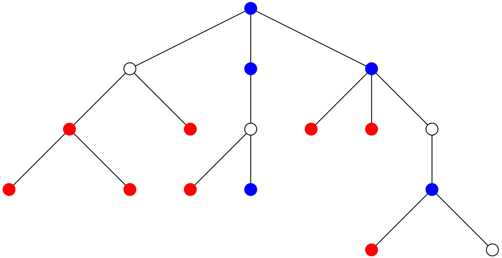

Matrix representation (blue red white):

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

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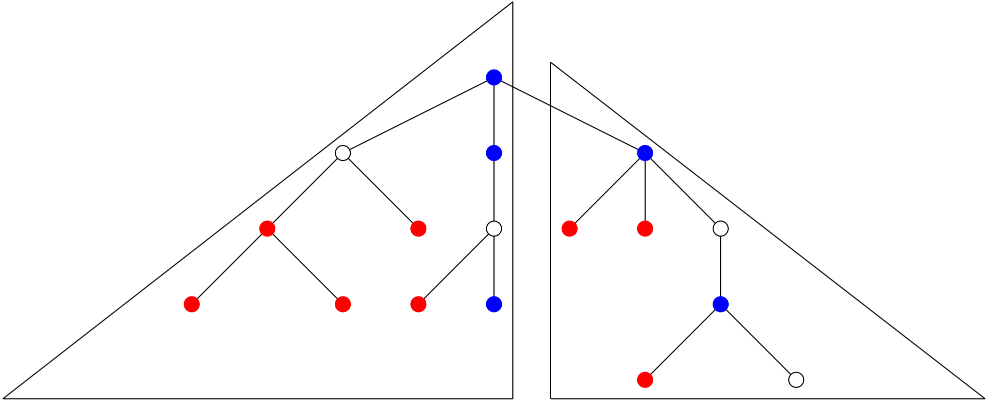

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Generating functions

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Corollary

$F_A^{(i)}(x)$ is always algebraic.

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n	1	2	3	4	5	6	7	8
$t_A^{(1)}(n)$	1	3	15	88	563	3812	26877	195349
$t_A^{(2)}(n)$	1	1	2	5	14	42	132	429
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- $F_A^{(1)}(x)^6 - 2F_A^{(1)}(x)^5 + (2x+1)F_A^{(1)}(x)^4 - 3xF_A^{(1)}(x)^3 + 5x^2F_A^{(1)}(x)^2 - x^2F_A^{(1)}(x) + x^3 = 0$
- $F_A^{(3)}(x)^6 - 2xF_A^{(3)}(x)^4 + xF_A^{(3)}(x)^3 + x^2F_A^{(3)}(x)^2 - x^2F_A^{(3)}(x) + x^3 = 0$

OEIS entry A381826

A381826 G.f. A(x) satisfies A(x) = C(x) / (1 - x*A(x)^2), where C(x) is the g.f. of A000108. 1

1, 2, 8, 41, 241, 1545, 10503, 74429, 543833, 4067510, 30985633, 239560975, 1874831287, 14823253892, 118222204539, 949963236834, 7683289712433, 62499664522578, 510992689465500, 4196824203859773, 34609480384100715, 286461380785102398, 2378954616256505177

(list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

LINKS Table of n, a(n) for n=0..22.

FORMULA a(n) = (1/(2*n+1)) * Sum_{k=0..n} binomial(2*n+1,k) * binomial(3*n-3*k,n-k).
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PROG (PARI) a(n) = sum(k=0, n, binomial(2*n+1, k)*binomial(3*n-3*k, n-k))/(2*n+1);

CROSSREFS Cf. A014137, A129442, A381827.
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Sequence in context: A389629 A381817 A294084 * A177340 A067119 A093935
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KEYWORD nonn

AUTHOR Seiichi Manyama, Mar 08 2025

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Straightforwardly shown to satisfy same functional equation as $F_A^{(3)}(x)$:

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- (If time) Bijective proof of one 2×2 result

Outline

- Preliminaries
- Catalog of 2×2 results
- Study of a 3×3 result
- Generalization to $n \times n$
- Related $n \times n$ family
- Same counting sequences?
- 2×2 bijective proof
- Summary

Catalog of counting sequences for 2×2 matrices

Matrix: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Sequences:

n	1	2	3	4	5	6	7	8	Formula
$t_A^{(1)}(n)$	1	0	0	0	0	0	0	0	0^{n-1}
$t_A^{(2)}(n)$	1	0	0	0	0	0	0	0	0^{n-1}
Total	2	0	0	0	0	0	0	0	$2 \cdot 0^{n-1}$

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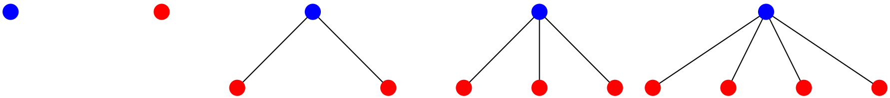

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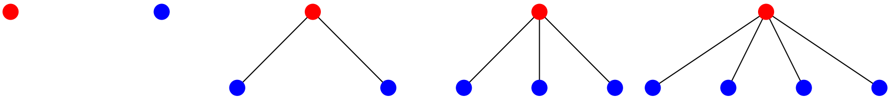

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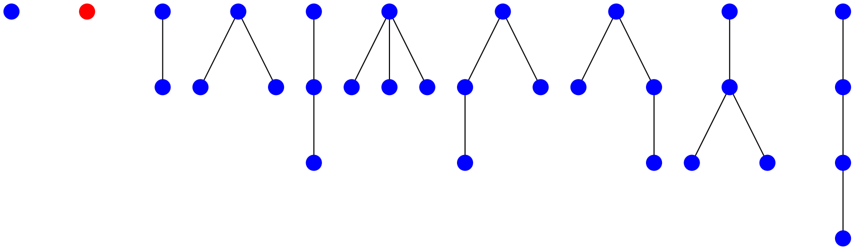

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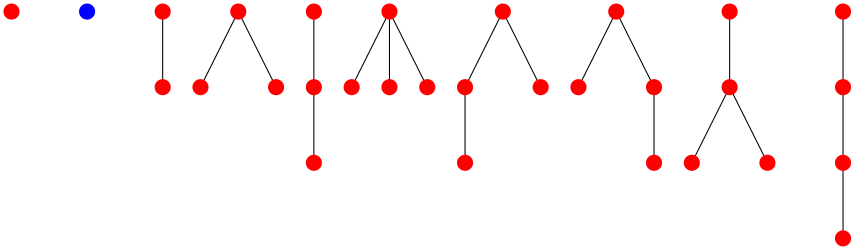

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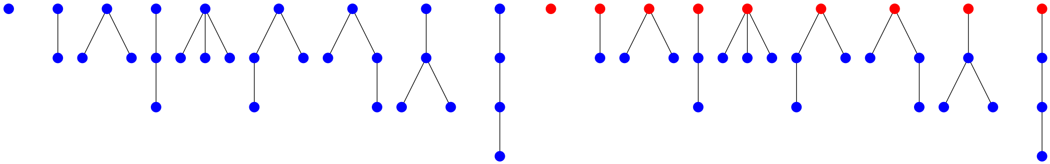

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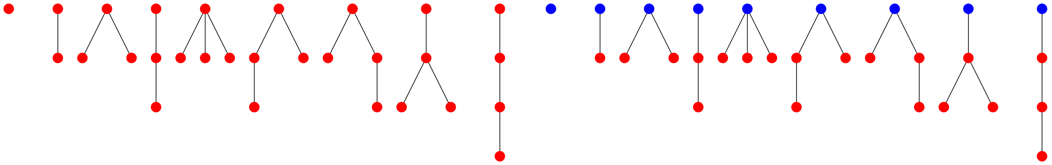

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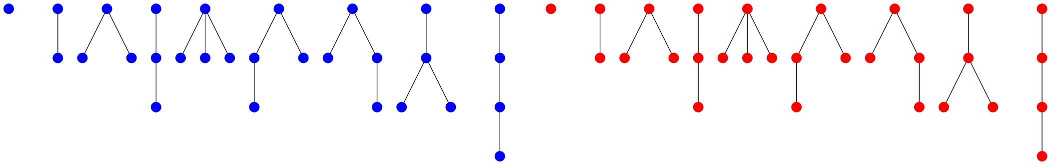

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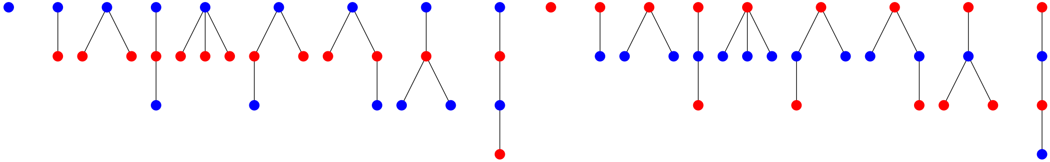
$t_A^{(2)}(n)$	1	1	2	5	14	42	132	429	C_{n-1}
Total	2	2	4	10	28	84	264	858	$2 \cdot C_{n-1}$



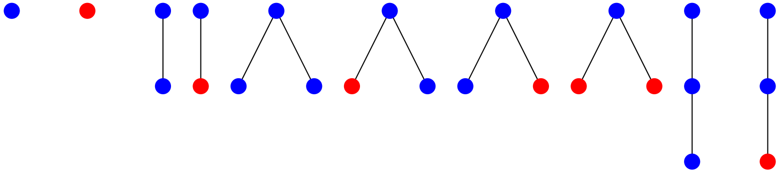
Catalog of counting sequences for 2×2 matrices

Matrix: $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Sequences:

n	1	2	3	4	5	6	7	8	Formula
$t_A^{(1)}(n)$	1	1	3	11	45	197	903	4279	$A001003(n-1)$
$t_A^{(2)}(n)$	1	1	3	11	45	197	903	4279	$A001003(n-1)$
Total	2	2	6	22	90	394	1806	8558	$2 \cdot A001003(n-1)$

A001003(n): Little Schröder numbers



Catalog of counting sequences for 2×2 matrices

Matrix: $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

Sequences:

n	1	2	3	4	5	6	7	8	Formula
$t_A^{(1)}(n)$	1	1	3	11	45	197	903	4279	$A001003(n-1)$
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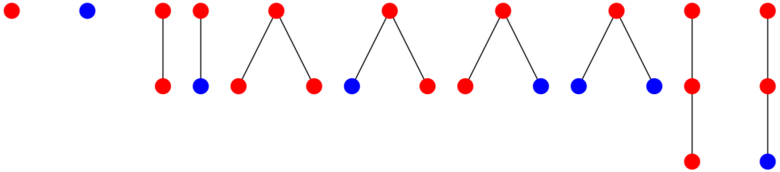


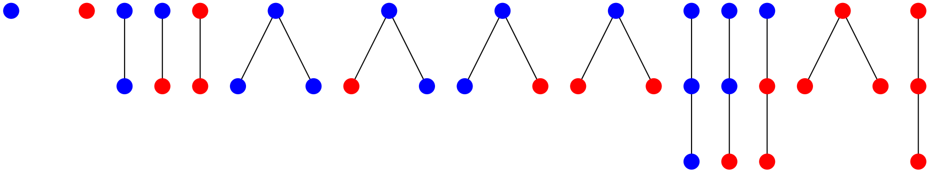
Catalog of counting sequences for 2×2 matrices

Matrix: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Sequences:

n	1	2	3	4	5	6	7	8	Formula
$t_A^{(1)}(n)$	1	2	7	29	131	625	3099	15818	$A007852(n)$
$t_A^{(2)}(n)$	1	1	2	5	14	42	132	429	C_{n-1}
Total	2	3	9	34	145	667	3231	16247	$A007852(n) + C_{n-1}$

$A007852(n)$: Number of antichains in rooted plane trees on n nodes



Catalog of counting sequences for 2×2 matrices

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Sequences:

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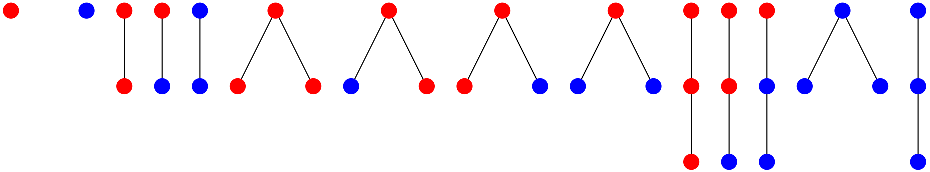


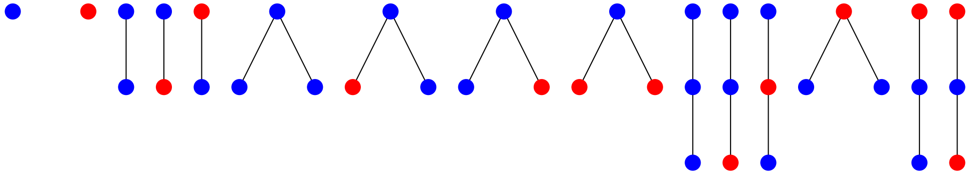
Catalog of counting sequences for 2×2 matrices

Matrix: $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Sequences:

n	1	2	3	4	5	6	7	8	Formula
$t_A^{(1)}(n)$	1	2	7	30	143	728	3876	21318	$\frac{1}{n}\binom{3n-2}{n-1}$
$t_A^{(2)}(n)$	1	1	3	12	55	273	1428	7752	$\frac{1}{2n-1}\binom{3n-3}{n-1}$
Total	2	3	10	42	198	1001	5304	29070	$\frac{2}{n}\binom{3n-3}{n-1}$

A006013$(n-1)$, A001764$(n-1)$, A007226$(n-1)$



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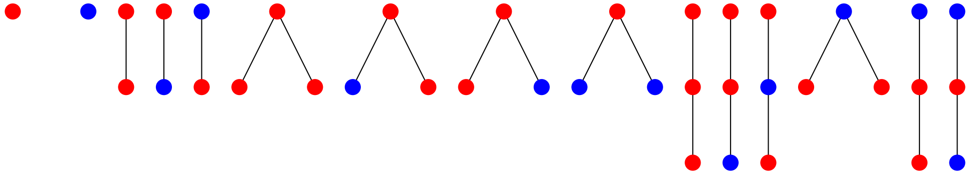
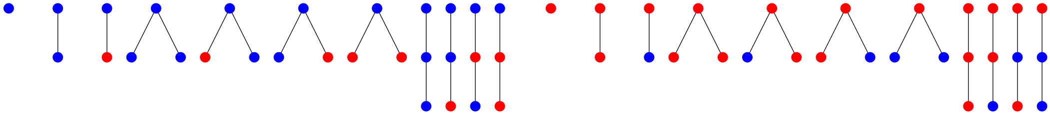


Catalog of counting sequences for 2×2 matrices

Matrix: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Sequences:

n	1	2	3	4	5	6	7	8	Formula
$t_A^{(1)}(n)$	1	2	8	40	224	1344	8448	54912	$2^{n-1}C_{n-1}$
$t_A^{(2)}(n)$	1	2	8	40	224	1344	8448	54912	$2^{n-1}C_{n-1}$
Total	2	4	16	80	448	2688	16896	109824	2^nC_{n-1}



Outline

- Preliminaries
- Catalog of 2×2 results
- Study of a 3×3 result
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An interesting 3×3 matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

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$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

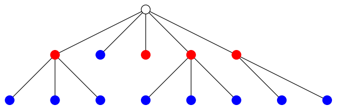
n	1	2	3	4	5	6	7	8	Formula
$t_A^{(1)}(n)$	1	0	0	0	0	0	0	0	0^{n-1}
$t_A^{(2)}(n)$	1	1	1	1	1	1	1	1	1
$t_A^{(3)}(n)$	1	2	5	13	34	89	233	610	
Total	3	3	6	14	35	90	234	611	

An interesting 3×3 matrix

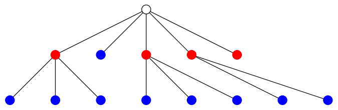
$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

n	1	2	3	4	5	6	7	8	Formula
$t_A^{(1)}(n)$	1	0	0	0	0	0	0	0	0^{n-1}
$t_A^{(2)}(n)$	1	1	1	1	1	1	1	1	1
$t_A^{(3)}(n)$	1	2	5	13	34	89	233	610	F_{2n-1}
Total	3	3	6	14	35	90	234	611	$0^{n-1} + 1 + F_{2n-1}$

Combinatorial proof that $t_A^{(3)}(n) = F_{2n-1}$



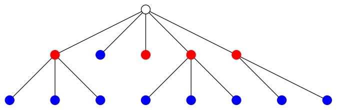
$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$



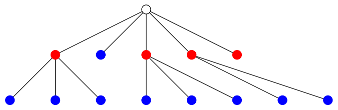
First Observation

This rule colors trees of height at most 2.

Combinatorial proof that $t_A^{(3)}(n) = F_{2n-1}$



$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$



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- Root can be any color (or specifically white for $t_A^{(3)}(n)$)

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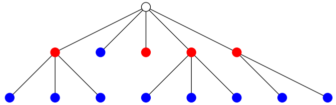
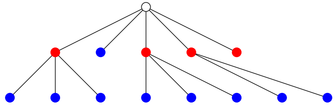


$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$



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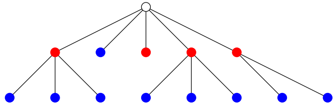
- Root can be any color (or specifically white for $t_A^{(3)}(n)$)
- Vertices at Level 1 can only be blue or red

Combinatorial proof that $t_A^{(3)}(n) = F_{2n-1}$



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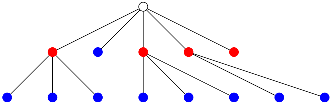
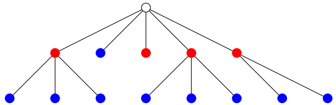


First Observation

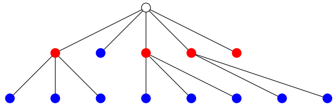
This rule colors trees of height at most 2.

- Root can be any color (or specifically white for $t_A^{(3)}(n)$)
- Vertices at Level 1 can only be blue or red
- Vertices at Level 2 must be blue leaves with parents colored red

Combinatorial proof that $t_A^{(3)}(n) = F_{2n-1}$

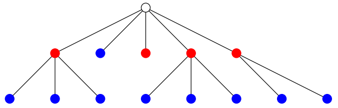


$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

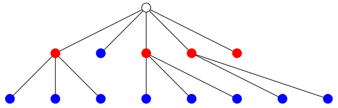


Given A-colored tree with n vertices and white root, look at rightmost leaf.

Combinatorial proof that $t_A^{(3)}(n) = F_{2n-1}$



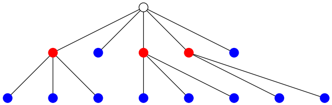
$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$



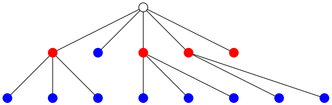
Given A-colored tree with n vertices and white root, look at rightmost leaf.

- It is either on Level 1 or 2.

Combinatorial proof that $t_A^{(3)}(n) = F_{2n-1}$



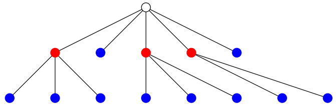
$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$



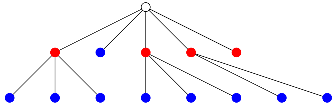
Given A-colored tree with n vertices and white root, look at rightmost leaf.

- It is either on Level 1 or 2.
- If it is on Level 1, it can be colored blue or red, and deleting it can leave behind any A-colored tree with $n-1$ vertices.

Combinatorial proof that $t_A^{(3)}(n) = F_{2n-1}$



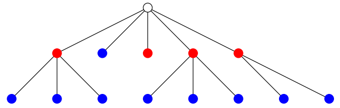
$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$



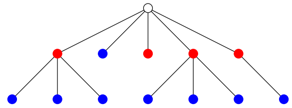
Given A-colored tree with n vertices and white root, look at rightmost leaf.

- It is either on Level 1 or 2.
- If it is on Level 1, it can be colored blue or red, and deleting it can leave behind any A-colored tree with $n-1$ vertices. $2t_A^{(3)}(n-1)$

Combinatorial proof that $t_A^{(3)}(n) = F_{2n-1}$



$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$



Given A-colored tree with n vertices and white root, look at rightmost leaf.

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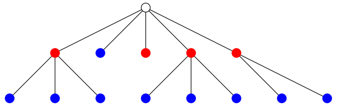
- If it is on Level 1, it can be colored blue or red, and deleting it can leave behind any A-colored tree with $n-1$ vertices. $2t_A^{(3)}(n-1)$
- If it is on Level 2, it must be colored blue, and deleting it can leave behind any A-colored tree with $n-1$ vertices *and rightmost leaf not blue on Level* 1.

Combinatorial proof that $t_A^{(3)}(n) = F_{2n-1}$



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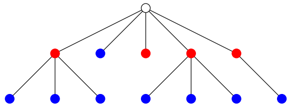

Given A-colored tree with n vertices and white root, look at rightmost leaf.

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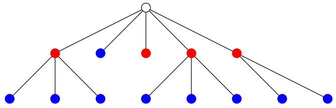
- If it is on Level 2, it must be colored blue, and deleting it can leave behind any A-colored tree with $n-1$ vertices *and rightmost leaf not blue on Level* 1. $t_A^{(3)}(n-1) - t_A^{(3)}(n-2)$

Combinatorial proof that $t_A^{(3)}(n) = F_{2n-1}$



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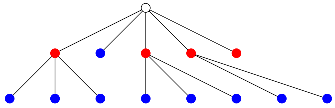
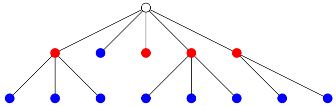


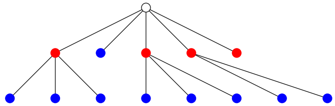
Theorem

$$t_A^{(3)}(n) = 3t_A^{(3)}(n-1) - t_A^{(3)}(n-2)$$

Combinatorial proof that $t_A^{(3)}(n) = F_{2n-1}$



$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$



Theorem

$$t_A^{(3)}(n) = 3t_A^{(3)}(n-1) - t_A^{(3)}(n-2)$$

Same recurrence as F_{2n-1}, and same initial conditions. $\square$

Outline

- Preliminaries
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Generalizing this example to more colors

$$A_m = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \qquad (m \times m)$$

Generalizing this example to more colors

$$A_m = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \qquad (m \times m)$$

- Colors trees of height at most m

Generalizing this example to more colors

$$A_m = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \qquad (m \times m)$$

- Colors trees of height at most m
- All subtrees colored according to $A_{m'}$ for some $m' < m$

Generalizing this example to more colors

$$A_m = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \qquad (m \times m)$$

- Colors trees of height at most m
- All subtrees colored according to $A_{m'}$ for some $m' < m$
- Rational generating functions $F_{A_m}^{(i)}(x)$ for all $1 \leq i \leq m$

Generalizing this example to more colors

$$A_m = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \qquad (m \times m)$$

- Colors trees of height at most m
- All subtrees colored according to $A_{m'}$ for some $m' < m$
- Rational generating functions $F_{A_m}^{(i)}(x)$ for all $1 \leq i \leq m$

Sequence of rational generating functions $F^{(1)}(x), F^{(2)}(x), F^{(3)}(x), F^{(4)}(x), \ldots$

What can we say about $F^{(i)}(x)$?

- $F^{(1)}(x) = x$

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- $F^{(4)}(x) = \dfrac{x-4x^2+4x^3-x^4}{1-7x+13x^2-7x^3+x^4}$

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- $F^{(4)}(x) = \dfrac{x-4x^2+4x^3-x^4}{1-7x+13x^2-7x^3+x^4}$
- $F^{(5)}(x) = \dfrac{x-11x^2+45x^3-88x^4+88x^5-45x^6+11x^7-x^8}{1-15x+83x^2-220x^3+303x^4-220x^5+83x^6-15x^7+x^8}$

What can we say about $F^{(i)}(x)$?

- $F^{(1)}(x) = x$
- $F^{(2)}(x) = \dfrac{x}{1-x}$
- $F^{(3)}(x) = \dfrac{x - x^2}{1 - 3x + x^2}$
- $F^{(4)}(x) = \dfrac{x - 4x^2 + 4x^3 - x^4}{1 - 7x + 13x^2 - 7x^3 + x^4}$
- $F^{(5)}(x) = \dfrac{x - 11x^2 + 45x^3 - 88x^4 + 88x^5 - 45x^6 + 11x^7 - x^8}{1 - 15x + 83x^2 - 220x^3 + 303x^4 - 220x^5 + 83x^6 - 15x^7 + x^8}$

Can we find a pattern in the numerators and denominators?

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- Coefficients of $p_i(x)$ and $q_i(x)$ alternate in sign and are unimodal.

But what *are* the coefficients?

What can we say about $F^{(i)}(x)$?

1,-15,83,-220,303 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,-15,83,-220,303**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A147990 Array A147985 (Polynomial coefficients) with zeros deleted. +30 6

1, 1, -1, 1, -3, 1, 1, -7, 13, -7, 1, **1, -15, 83, -220, 303**, -220, 83, -15, 1, 1, -31, 413, -3141, 15261, -50187, 115410, -189036, 222621, -189036, 115410, -50187, 15261, -3141, 413, -31, 1, 1, -63, 1839, -33150, 414861, -3841195, 27378213, -154299168

(list; table; graph; refs; listen; history; text; internal format)

OFFSET	1,5
LINKS	Table of n, a(n) for n=1..45. Clark Kimberling, Polynomials associated with reciprocation, JIS 12 (2009) 09.3.4
FORMULA	Let s(1)=x and for n>=2, let s(n)=s(n,x)=S(n,y), where y=x^(1/2) and S(n,x) is as at A147985. Then A147990 gives the coefficients of the polynomials s(n).
EXAMPLE	s(1)=x s(2)=S(2,y)=x-1 s(3)=S(3,y)=x^2-3*x+1 s(4)=S(4,y)=x^4-7*x^3+13*x^2-7*x+1 so that as an array A147990 begins with 1 1 -1 1 -3 1 1 -7 13 -7 1
CROSSREFS	Cf. A147985, A147986, A147987, A147988, A147989, A147991, A147992, A147993.
KEYWORD	sign,tabl
AUTHOR	Clark Kimberling, Nov 25 2008
STATUS	approved

page 1

What can we say about $F^{(i)}(x)$?

A147985 Coefficients of numerator polynomials S(n,x) associated with reciprocation. 7

1, 0, 1, 0, -1, 1, 0, -3, 0, 1, 1, 0, -7, 0, 13, 0, -7, 0, 1, 1, 0, -15, 0, 83, 0, -220, 0, 303, 0, -220, 0, 83, 0, -15, 0, 1, 1, 0, -31, 0, 413, 0, -3141, 0, 15261, 0, -50187, 0, 115410, 0, -189036, 0, 222621, 0, -189036, 0, 115410, 0, -50187, 0, 15261, 0, -3141, 0, 413, 0

(list; graph; refs; listen; history; text; internal format)

OFFSET 1,8

COMMENTS
1. S(n)=U(n-1)V(n-1) where U(n-1)=S(n-1)+S(1)*S(2)*...*S(n-2) and V(n-1)=S(n-1)-S(1)*S(2)*...*S(n-2), for n>=2. If U(n) and V(n) are written as polynomials U(n,x) and V(n,x), then V(n,x)=U(n,-x). See A147989 for coefficients of U(n).
2. S(n)=S(n-1)^2+S(n-1)*S(n-2)^2-S(n-2)^4 for n>2. (The Gorskov-Wirsting polynomials also have this recurrence; see H. L. Montgomery, Ten Lectures on the Interface between Analytic Number Theory and Harmonic Analysis, CBMS Regional Conference Series in Mathematics, 84, AMS, pp. 183-190.)
3. For n>0, the 2^(n-1) zeros of S(n) are real. If r is a zero of S(n), then -r and 1/r are zeros of S(n).
4. If r is a zero of S(n), then the numbers z satisfying r=z-1/z and r=z+1/z are zeros of S(n+1).
5. If n>2, then S(n,1)=1 and S(n,2)=A127814(n).
6. S(n,2^(1/2))=-1 for n>2 and S(n,2^(-1/2))=-2^(1-n) for n>1.

LINKS
Peter J. C. Moses, Rows n = 1..13 of irregular triangle, flattened
Clark Kimberling, Polynomials associated with reciprocation, Journal of Integer Sequences 12 (2009, Article 09.3.4) 1-11.

FORMULA
The basic idea is to iterate the reciprocation-difference mapping x/y -> x/y-y/x.
Let x be an indeterminate, S(1)=x, T(1)=1 and for n>1, define S(n)=S(n-1)^2-T(n-1)^2 and T(n)=S(n-1)*T(n-1), so that S(n)/T(n)=S(n-1)/T(n-1)-T(n-1)/S(n-1).

EXAMPLE
S(1)=x
S(2)=x^2-1=(x-1)(x+1)
S(3)=x^4-3*x^2+1=(x^2+x-1)(x^2-x-1)
S(4)=x^8-7*x^6+13*x^4-7*x^2+1=(x^4+x^3-3*x^2-x+1)(x^4-x^3-3*x^2+x+1),
so that, as an array, sequence begins with
1 0
1 0 -1
1 0 -3 0 1
1 0 -7 0 13 0 -7 0 1

What can we say about $F^{(i)}(x)$?

A147986 Coefficients of denominator polynomials T(n,x) associated with reciprocation. 6

1, 1, 0, 1, 0, -1, 0, 1, 0, -4, 0, 4, 0, -1, 0, 1, 0, -11, 0, 45, 0, -88, 0, 88, 0, -45, 0, 11, 0, -1, 0, 1, 0, -26, 0, 293, 0, -1896, 0, 7866, 0, -22122, 0, 43488, 0, -60753, 0, 60753, 0, -43488, 0, 22122, 0, -7866, 0, 1896, 0, -293, 0, 26, 0, -1, 0, 1, 0, -57, 0, 1512, 0

(list; graph; refs; listen; history; text; internal format)

OFFSET 1,10

COMMENTS T(n)=S(1)*S(2)*...*S(n-1). The degree of S(n) in x is m=2^(n-1), so that the degree of T(n) is m-1. Write the zeros of T(n) as r(1)<r(2)<...<r(m-1) and the zeros of S(n) as z(1)<z(2)<...<z(m). Then z(1)<r(1)<z(2)<r(2)<...<r(m-1)<z(m); i.e., the zeros of T(n) intersperse the zeros of S(n).

LINKS Table of n, a(n) for n=1..69.
Clark Kimberling, Polynomials associated with reciprocation, Journal of Integer Sequences 12 (2009, Article 09.3.4) 1-11.

FORMULA The basic idea is to iterate the reciprocation-difference mapping x/y -> x/y-y/x.
Let x be an indeterminate, S(1)=x, T(1)=1 and for n>1, define S(n)=S(n-1)^2-T(n-1)^2 and T(n)=S(n-1)*T(n-1), so that S(n)/T(n)=S(n-1)/T(n-1)-T(n-1)/S(n-1).

EXAMPLE
```
T(1) = 1
T(2) = x
T(3) = x^3-x
T(4) = x^7-4*x^5+4*x^3-x
so that, as an array, the sequence begins with:
1
1 0
1 0 -1 0
1 0 -4 0 4 0 -1 0
```

Studying A147985 and A147986

A147985: Coefficients of numerator polynomials $S_n(x)$ associated with reciprocation.

A147986: Coefficients of denominator polynomials $T_n(x)$ associated with reciprocation.

- Start with $\frac{x}{1}$. Iterate $\frac{S_n(x)}{T_n(x)} = \frac{S_{n-1}(x)}{T_{n-1}(x)} - \frac{T_{n-1}(x)}{S_{n-1}(x)}$

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- Recurrences: $S_n(x) = S_{n-1}(x)^2 - T_{n-1}(x)^2$; $T_n(x) = S_{n-1}(x)T_{n-1}(x)$

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A147986: Coefficients of denominator polynomials $T_n(x)$ associated with reciprocation.

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- $S_n(x)$ has the same coefficients as $q_n(x)$, but it's even and has degree 2^{n-1} instead of 2^{n-2}.

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- Recurrences: $S_n(x) = S_{n-1}(x)^2 - T_{n-1}(x)^2$; $T_n(x) = S_{n-1}(x)T_{n-1}(x)$
- $S_n(x)$ has the same coefficients as $q_n(x)$, but it's even and has degree 2^{n-1} instead of 2^{n-2}.
- $T_n(x)$ has the opposite coefficients as $p_n(x)$, plus it's odd and has degree $2^{n-1} - 1$ instead of 2^{n-2}.

What can we say about $F^{(i)}(x)$?

Theorem

$$p_i(x) = \begin{cases} x & i = 1 \\ p_{i-1}(x) q_{i-1}(x) & i > 1 \end{cases}$$

$$q_i(x) = \begin{cases} 1 & i = 1 \\ q_{i-1}(x)^2 - \frac{p_{i-1}(x)^2}{x} & i > 1 \end{cases}$$

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Corollary

For $i \geq 3$, $p_i(x^2) = -xT_i(x)$ and $q_i(x^2) = S_i(x)$.

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Proof of Theorem follows by induction and manipulation of generating functions.

Outline

- Preliminaries
- Catalog of 2×2 results
- Study of a 3×3 result
- Generalization to $n \times n$
- Related $n \times n$ family
- Same counting sequences?
- 2×2 bijective proof
- Summary

Another, seemingly less interesting 3×3 matrix

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

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n	1	2	3	4	5	6	7	8	Formula
$t_B^{(1)}(n)$	1	1	2	5	14	42	132	429	C_{n-1}
$t_B^{(2)}(n)$	1	2	7	29	131	625	3099	15818	$A007852(n)$
$t_B^{(3)}(n)$	1	3	15	87	544	3566	24165	167904	
Total	3	6	24	121	689	4233	27396	184151	

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Total	3	6	24	121	689	4233	27396	184151	$***$

$***$ Total $= C_{n-1} + A007852(n) + A394121(n)$

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Total	3	6	24	121	689	4233	27396	184151	$***$

$***$ Total $= C_{n-1} + A007852(n) + A394121(n)$ Functional Equation:

$$F_B^{(3)}(x)^8 - F_B^{(3)}(x)^7 + 7xF_B^{(3)}(x)^6 - 4xF_B^{(3)}(x)^5 + 13x^2F_B^{(3)}(x)^4 - 4x^2F_B^{(3)}(x)^3 + 7x^3F_B^{(3)}(x)^2 - x^3F_B^{(3)}(x) + x^4 = 0$$

Generalizing this example to more colors

$$B_m = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \qquad (m \times m)$$

Generalizing this example to more colors

$$B_m = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \qquad (m \times m)$$

- Colors trees of any height

Generalizing this example to more colors

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- Colors trees of any height
- All subtrees colored according to $B_{m'}$ for some $m' \leq m$

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- Colors trees of any height
- All subtrees colored according to $B_{m'}$ for some $m' \leq m$
- Generating functions $F_{B_m}^{(i)}(x)$ for all $1 \leq i \leq m$, not rational this time

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- All subtrees colored according to $B_{m'}$ for some $m' \leq m$
- Generating functions $F_{B_m}^{(i)}(x)$ for all $1 \leq i \leq m$, not rational this time

Sequence of generating functions $F^{(1)}(x), F^{(2)}(x), F^{(3)}(x), F^{(4)}(x), \ldots$

What can we say about $F^{(i)}(x)$?

- $F^{(1)}(x)^2 - F^{(1)}(x) + x = 0$

What can we say about $F^{(i)}(x)$?

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- $F^{(2)}(x)^4 - F^{(2)}(x)^3 + 3xF^{(2)}(x)^2 - xF^{(2)}(x) + x^2 = 0$

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- $F^{(2)}(x)^4 - F^{(2)}(x)^3 + 3xF^{(2)}(x)^2 - xF^{(2)}(x) + x^2 = 0$
- $F^{(3)}(x)^8 - F^{(3)}(x)^7 + 7xF^{(3)}(x)^6 - 4xF^{(3)}(x)^5 + 13x^2F^{(3)}(x)^4 - 4x^2F^{(3)}(x)^3 + 7x^3F^{(3)}(x)^2 - x^3F^{(3)}(x) + x^4 = 0$

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- $F^{(3)}(x)^8 - F^{(3)}(x)^7 + 7xF^{(3)}(x)^6 - 4xF^{(3)}(x)^5 + 13x^2F^{(3)}(x)^4 - 4x^2F^{(3)}(x)^3 + 7x^3F^{(3)}(x)^2 - x^3F^{(3)}(x) + x^4 = 0$

Can we find a pattern in functional equations?

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- $F^{(1)}(x)^2 - F^{(1)}(x) + x = 0$
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Can we find a pattern in functional equations?

- Degree for $F^{(i)}(x)$ is 2^i.

What can we say about $F^{(i)}(x)$?

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- $F^{(3)}(x)^8 - F^{(3)}(x)^7 + 7xF^{(3)}(x)^6 - 4xF^{(3)}(x)^5 + 13x^2F^{(3)}(x)^4 - 4x^2F^{(3)}(x)^3 + 7x^3F^{(3)}(x)^2 - x^3F^{(3)}(x) + x^4 = 0$

Can we find a pattern in functional equations?

- Degree for $F^{(i)}(x)$ is 2^i.
- Integer coefficients are palindromic and alternate in sign.

What can we say about $F^{(i)}(x)$?

- $F^{(1)}(x)^2 - F^{(1)}(x) + x = 0$
- $F^{(2)}(x)^4 - F^{(2)}(x)^3 + 3xF^{(2)}(x)^2 - xF^{(2)}(x) + x^2 = 0$
- $F^{(3)}(x)^8 - F^{(3)}(x)^7 + 7xF^{(3)}(x)^6 - 4xF^{(3)}(x)^5 + 13x^2F^{(3)}(x)^4 - 4x^2F^{(3)}(x)^3 + 7x^3F^{(3)}(x)^2 - x^3F^{(3)}(x) + x^4 = 0$

Can we find a pattern in functional equations?

- Degree for $F^{(i)}(x)$ is 2^i.
- Integer coefficients are palindromic and alternate in sign.
- Coefficient on $F^{(i)}(x)^j$ includes $x^{\left\lfloor \frac{2^i - j}{2} \right\rfloor}$.

What can we say about $F^{(i)}(x)$?

- $F^{(1)}(x)^2 - F^{(1)}(x) + x = 0$
- $F^{(2)}(x)^4 - F^{(2)}(x)^3 + 3xF^{(2)}(x)^2 - xF^{(2)}(x) + x^2 = 0$
- $F^{(3)}(x)^8 - F^{(3)}(x)^7 + 7xF^{(3)}(x)^6 - 4xF^{(3)}(x)^5 + 13x^2F^{(3)}(x)^4 - 4x^2F^{(3)}(x)^3 + 7x^3F^{(3)}(x)^2 - x^3F^{(3)}(x) + x^4 = 0$

Can we find a pattern in functional equations?

- Degree for $F^{(i)}(x)$ is 2^i.
- Integer coefficients are palindromic and alternate in sign.
- Coefficient on $F^{(i)}(x)^j$ includes $x^{\left\lfloor \frac{2^i - j}{2} \right\rfloor}$.

But what *are* the integer coefficients?

What can we say about $F^{(i)}(x)$?

- $F_{B_m}^{(1)}(x)^2 - F_{B_m}^{(1)}(x) + x = 0$
- $F_{B_m}^{(2)}(x)^4 - F_{B_m}^{(2)}(x)^3 + 3xF_{B_m}^{(2)}(x)^2 - xF_{B_m}^{(2)}(x) + x^2 = 0$
- $F_{B_m}^{(3)}(x)^8 - F_{B_m}^{(3)}(x)^7 + 7xF_{B_m}^{(3)}(x)^6 - 4xF_{B_m}^{(3)}(x)^5 + 13x^2F_{B_m}^{(3)}(x)^4 - 4x^2F_{B_m}^{(3)}(x)^3 + 7x^3F_{B_m}^{(3)}(x)^2 - x^3F_{B_m}^{(3)}(x) + x^4 = 0$

- $F_{A_m}^{(2)}(x) = \dfrac{x}{1-x}$
- $F_{A_m}^{(3)}(x) = \dfrac{x-x^2}{1-3x+x^2}$
- $F_{A_m}^{(4)}(x) = \dfrac{x-4x^2+4x^3-x^4}{1-7x+13x^2-7x^3+x^4}$

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- $F^{(2)}_{A_m}(x) = \dfrac{x}{1-x}$
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These are the same numbers, just interleaved and with different signs!

What can we say about $F^{(i)}(x)$?

A147987: Coefficients of numerator polynomials $P_n(x)$ associated with reciprocation.

A147988: Coefficients of denominator polynomials $Q_n(x)$ associated with reciprocation.

- Start with $\frac{x}{1}$. Iterate $\frac{P_n(x)}{Q_n(x)} = \frac{P_{n-1}(x)}{Q_{n-1}(x)} + \frac{Q_{n-1}(x)}{P_{n-1}(x)}$

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- P_n is like S_n but has all positive coefficients; similar for Q_n and T_n.
- Coefficients of functional equation interleave coefficients of $Q_n(x)$ with those of $-P_n(x)$ while also multiplying by a power of x.

What can we say about $F^{(i)}(x)$?

Theorem

$$x^{2^{i-1}}\left(\frac{1}{\sqrt{x}}Q_{i+1}\left(\frac{F^{(i)}(x)}{\sqrt{x}}\right)-P_{i+1}\left(\frac{F^{(i)}(x)}{\sqrt{x}}\right)\right)=0$$

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Proof of Theorem follows by induction and manipulation of generating functions, though it's somewhat more cumbersome than for A_m.

Outline

- Preliminaries
- Catalog of 2×2 results
- Study of a 3×3 result
- Generalization to $n \times n$
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Definition

Matrices A and B are strictly tree coloring equivalent if they have identical sequences $t_A^{(i)}(n)$ and $t_B^{(i)}(n)$ for all colors i.

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Definition

For a matrix A, colors i and j are interchangeable if sequences $t_A^{(i)}(n)$ and $t_A^{(j)}(n)$ are identical.

Different coloring rules with same counting sequences?

Theorem

Suppose i and j are interchangeable colors in A. Let ℓ be such that $a_{\ell i} = 0$ and $a_{\ell j} = 1$. Swapping these gives a strictly tree coloring equivalent matrix.

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Example

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Colors 2 and 3 are interchangeable (both Catalan)

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$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & \color{red}{0} & \color{red}{1} \end{bmatrix}$$

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- $a_{32} = 0$ and $a_{33} = 1$ ($\ell = 3$)

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Example

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- $a_{32} = 0$ and $a_{33} = 1$ ($\ell = 3$)
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$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & \color{red}{0} & \color{red}{1} \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- Colors 2 and 3 are interchangeable (both Catalan)
- $a_{32} = 0$ and $a_{33} = 1$ ($\ell = 3$)
- A and B are strictly tree coloring equivalent.
- Can get others with more theorem applications.

Different coloring rules with same counting sequences?

Theorem

Suppose i and j are interchangeable colors in A. Let ℓ be such that $a_{\ell i} = 0$ and $a_{\ell j} = 1$. Swapping these gives a strictly tree coloring equivalent matrix.

Question

If A and B are strictly tree coloring equivalent, can B always be obtained from A by repeated applications of this theorem?

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No!

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$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Matrices with same *total* counting sequences?

Definition

$m \times m$ matrices A and B are tree coloring equivalent if the sequences $\sum_{i=1}^{m} t_A^{(i)}(n)$ and $\sum_{i=1}^{m} t_B^{(i)}(n)$ are identical.

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If A and B are tree coloring equivalent, is B always a permutation of a matrix that's strictly tree coloring equivalent with A?

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2×2: YES

Matrices with same *total* counting sequences?

Question

If A and B are tree coloring equivalent, is B always a permutation of a matrix that's strictly tree coloring equivalent with A?

Answer

2×2: YES

3×3: NO

Example

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \Bigg| \quad B_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

First pair of tree coloring equivalent matrices

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

n	1	2	3	4	5	6	7	8	Formula
$t_{A_1}^{(1)}(n)$	1	0	0	0	0	0	0	0	0^{n-1}
$t_{A_1}^{(2)}(n)$	1	2	6	22	90	394	1806	8558	$A006318(n-1)$
$t_{A_1}^{(3)}(n)$	1	2	6	22	90	394	1806	8558	$A006318(n-1)$
Total	3	4	12	44	180	788	3612	17116	$2 \cdot A006318(n-1) + 0^{n-1}$

- A001003(n): Little Schröder numbers
- A006318(n): Big Schröder numbers

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n	1	2	3	4	5	6	7	8	Formula
$t_{A_2}^{(1)}(n)$	1	2	6	22	90	394	1806	8558	$A006318(n-1)$
$t_{A_2}^{(2)}(n)$	1	1	3	11	45	197	903	4279	$A001003(n-1)$
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Total	3	4	12	44	180	788	3612	17116	$2 \cdot A006318(n-1) + 0^{n-1}$

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Second pair of tree coloring equivalent matrices

$$B_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

n	1	2	3	4	5	6	7	8	Formula
$t_{B_1}^{(1)}(n)$	1	3	14	75	434	2646	16764	109395	$(2^n - 1)\, C_{n-1}$
$t_{B_1}^{(2)}(n)$	1	1	2	5	14	42	132	429	C_{n-1}
$t_{B_1}^{(3)}(n)$	1	1	2	5	14	42	132	429	C_{n-1}
Total	3	5	18	85	462	2730	17028	110253	$(2^n + 1)\, C_{n-1}$

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n	1	2	3	4	5	6	7	8	Formula
$t_{B_2}^{(1)}(n)$	1	2	8	40	224	1344	8448	54912	$2^{n-1} C_{n-1}$
$t_{B_2}^{(2)}(n)$	1	2	8	40	224	1344	8448	54912	$2^{n-1} C_{n-1}$
$t_{B_2}^{(3)}(n)$	1	1	2	5	14	42	132	429	C_{n-1}
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Wrap-up of (strict) tree coloring equivalence

Theorem

Suppose i and j are interchangeable colors in A. Let ℓ be such that $a_{\ell i} = 0$ and $a_{\ell j} = 1$. Swapping these gives a strictly tree coloring equivalent matrix.

- There's a generalization where, if A and B are strictly tree coloring equivalent, then B can always be obtained from A by that generalization.

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Suppose i and j are interchangeable colors in A. Let ℓ be such that $a_{\ell i} = 0$ and $a_{\ell j} = 1$. Swapping these gives a strictly tree coloring equivalent matrix.

- There's a generalization where, if A and B are strictly tree coloring equivalent, then B can always be obtained from A by that generalization.
- Counterexample for above theorem built with B_1 and B_2 from previous slide

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

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Deep dive into no red-followed-by-red case

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$: Trees colored blue and red where red can't follow red

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- $t_A^{(1)}(n) = A006013(n-1) = \frac{1}{n}\binom{3n-2}{n-1} = \frac{(3n-2)!}{n!(2n-1)!}$
- $t_A^{(2)}(n) = A001764(n-1) = \frac{1}{2n-1}\binom{3n-3}{n-1} = \frac{(3n-3)!}{(n-1)!(2n-1)!}$
- Total $= A007226(n-1) = \frac{2}{n}\binom{3n-3}{n-1} = \frac{2\cdot(3n-3)!}{n!(2n-2)!}$

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Follows from

Theorem (Kirschenhofer, Prodinger, Tichy)

Total number of independent sets over all plane trees with n vertices: $\frac{2}{n}\binom{3n-3}{n-1}$

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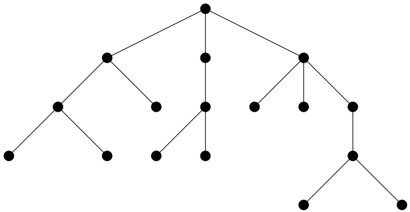
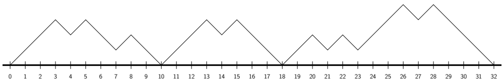
We give another bijective proof.

The “Glove Bijection”

Bijection between plane trees with n vertices and Dyck paths $(0, 0)$ to $(2n - 2, 0)$





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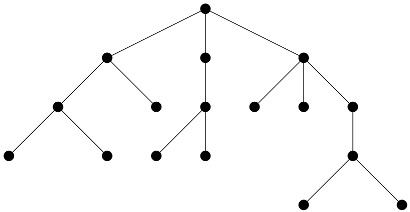

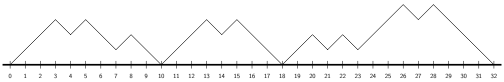

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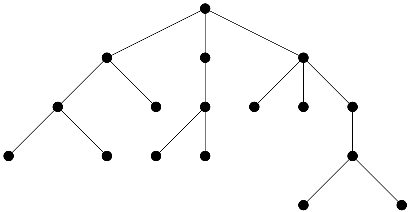

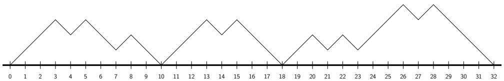

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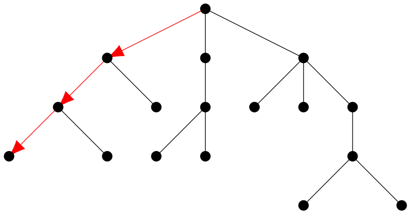

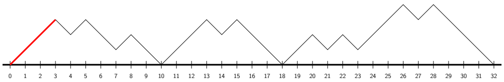

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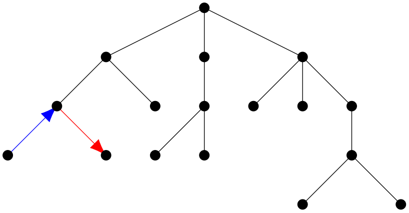

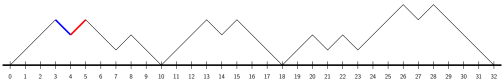

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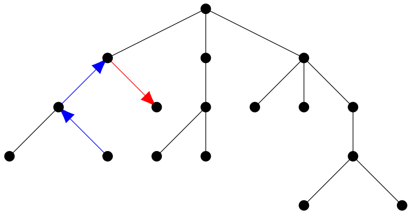
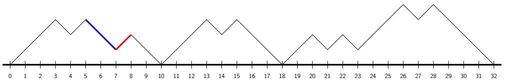
  - Up in tree $\leftrightarrow$ D in path





Relevant comment on A001764

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The number of lattice paths from (0,0) to (2n,0) using an
Up-step=(1,1) and a Down-step=(0,-2) and staying above the
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Knowing this, not hard to interpret A006013 as number of Dyck paths $(0,0)$ to $(4n+2,0)$, all but first ascent even length

Bijection for no red-followed-by-red case

Goal: Bijection between n-vertex trees colored $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and Dyck paths $(0, 0)$ to $(4n - 4, 0)$ or $(4n - 2, 0)$, all ascents even length except possibly first one

- Blue root $\leftrightarrow$ Path $(0, 0)$ to $(4n - 2, 0)$
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Intermediate objects: Trees with even downward path property

Even downward path property

Definition

In a tree, a downward path is any path that starts at a vertex with at least two children, progresses one step downward to a non-leftmost child, and then proceeds from there via leftmost children to a leaf.

Even downward path property

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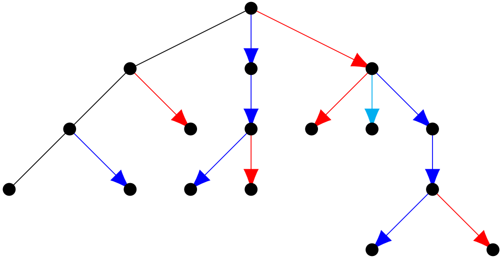
- Exactly one downward path ending in each non-leftmost leaf
- Each edge in exactly one downward path unless leftmost descendant of root



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Tree has even downward path property if all downward paths have even length

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Definition

$\mathcal{E}_k :=$ Set of plane trees with k vertices and even downward path property

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$\mathcal{E}_k \leftrightarrow$ Path $(0, 0)$ to $(2k - 2, 0)$: Glove Bijection

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$\mathcal{E}_k \leftrightarrow$ Path $(0,0)$ to $(2k-2,0)$: Glove Bijection
A-colored tree $\leftrightarrow \mathcal{E}_k$: Recursive construction

Colored tree to $\mathcal{E}_k$

Each vertex (except a red root) $\leftrightarrow$ two-vertex chain

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Base Cases:

Colored tree to $\mathcal{E}_k$

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Base Cases:

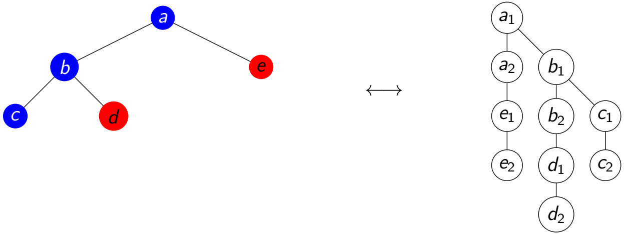
Recursive Cases: Delete rightmost leaf, apply bijection, then add 2 vertices

Analysis of recursive cases

Recursive Cases: Delete rightmost leaf, apply bijection, then add 2 vertices

Case 1: Deleted vertex or its parent is red

Attach new chain below bottom of parent's chain

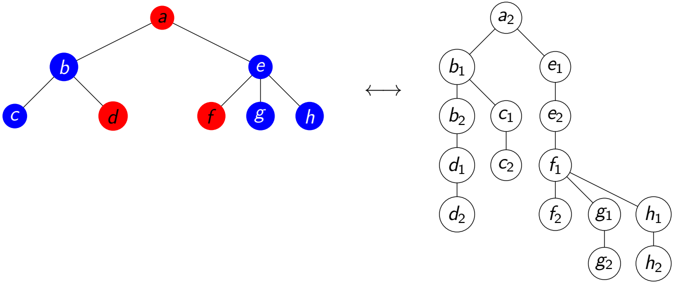


Analysis of recursive cases

Recursive Cases: Delete rightmost leaf, apply bijection, then add 2 vertices

Case 1: Deleted vertex or its parent is red
Attach new chain below bottom of parent's chain

Case 2: Both blue; deleted vertex has a red left-sibling
Attach new chain below top of rightmost red left-sibling's chain



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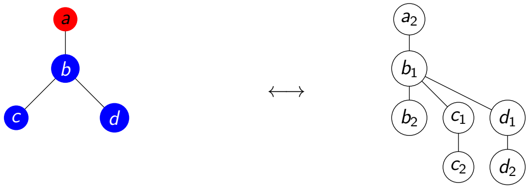
Case 3: Both blue; deleted vertex has no red left-sibling
Attach new chain below top of parent's chain



$\mathcal{E}_k$ to colored tree

Two-vertex chain $\leftrightarrow$ each vertex (except a red root)

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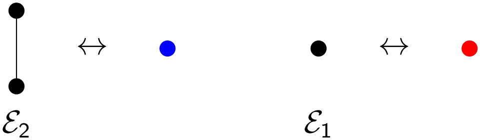

$\mathcal{E}_k$ to colored tree

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Base Cases:

$\mathcal{E}_2$ $\qquad$ $\mathcal{E}_1$

Recursive Cases: Delete rightmost leaf and its parent, apply bijection, then add colored vertex

$\mathcal{E}_k$ to colored tree

Two-vertex chain $\leftrightarrow$ each vertex (except a red root)

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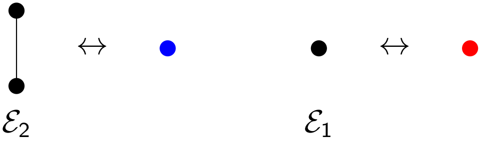


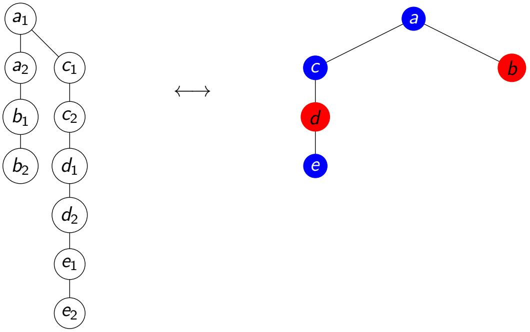
Recursive Cases: Delete rightmost leaf and its parent, apply bijection, then add colored vertex

Parent of rightmost leaf has only one child: Even downward path property

Analysis of recursive cases for inverse

Recursive Cases: Delete rightmost leaf and its parent, apply bijection, then add colored vertex

Case 1: Deleted chain below bottom of parent chain
New rightmost child of parent chain's vertex, opposite color

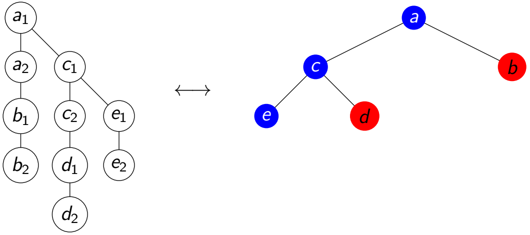


Analysis of recursive cases for inverse

Recursive Cases: Delete rightmost leaf and its parent, apply bijection, then add colored vertex

Case 1: Deleted chain below bottom of parent chain
New rightmost child of parent chain's vertex, opposite color

Case 2: Deleted chain below top of parent chain; parent chain's vertex blue
New blue child of parent chain's vertex left of leftmost red child



Analysis of recursive cases for inverse

Recursive Cases: Delete rightmost leaf and its parent, apply bijection, then add colored vertex

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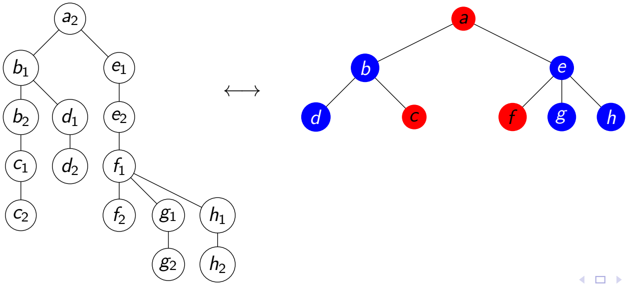
New blue child of parent chain's vertex left of leftmost red child

Case 3: Deleted chain below top of parent chain; parent chain's vertex red
New rightmost child of *parent of* parent chain's vertex, color blue



Outline

- Preliminaries
- Catalog of 2×2 results
- Study of a 3×3 result
- Generalization to $n \times n$
- Related $n \times n$ family
- Same counting sequences?
- 2×2 bijective proof
- Summary

Summary of findings

Paper available on ArXiv

- Catalogs all coloring rules with 2 and 3 colors

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- Catalogs all coloring rules with 2 and 3 colors
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Summary of findings

Paper available on ArXiv

- Catalogs all coloring rules with 2 and 3 colors
- Connects 3-color rules with existing OEIS sequences
- Contains several bijections and combinatorial proofs, including the ones from this talk
- Includes the material about (strict) tree coloring equivalence
- Discusses several infinite families of coloring rules
    - Includes the two examples from this talk

Summary of findings

Paper available on ArXiv

- Catalogs all coloring rules with 2 and 3 colors
- Connects 3-color rules with existing OEIS sequences
- Contains several bijections and combinatorial proofs, including the ones from this talk
- Includes the material about (strict) tree coloring equivalence
- Discusses several infinite families of coloring rules
    - Includes the two examples from this talk
    - Also includes an infinite family of matrices that are tree coloring equivalent but not strictly tree coloring equivalent

Thank you!