



Colgate University

A Computational Approach to Improving Bounds on the Hales–Jewett Numbers

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Rutgers Experimental Mathematics Seminar

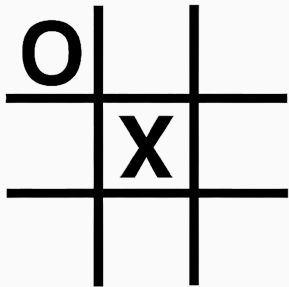
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Hales–Jewett Numbers

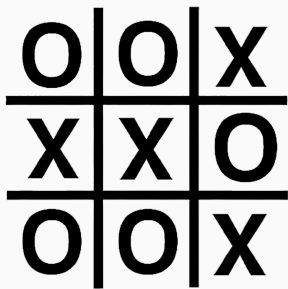
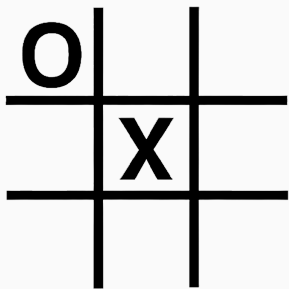
An Illustrative Example: High-Dimensional Tic-Tac-Toe



A Simple Question

Can a fully filled tic-tac-toe board avoid a winning line?

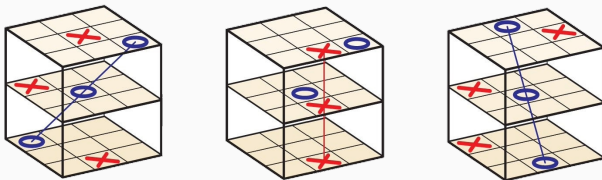
An Illustrative Example: High-Dimensional Tic-Tac-Toe



A Simple Question

Can a fully filled tic-tac-toe board avoid a winning line?

An Illustrative Example: High-Dimensional Tic-Tac-Toe



A Less Trivial Question

Can a fully filled 3D tic-tac-toe board avoid a winning line?

Why This Is Hard

The 3×3 cube admits $2^{27} = 134,217,728$ fully filled boards.

Informal Statement of the Hales–Jewett Theorem

Question. Is there a game board dimension large enough that a fully filled board cannot avoid a winning line?

Informal Statement of the Hales–Jewett Theorem

Question. Is there a game board dimension large enough that a fully filled board cannot avoid a winning line?

The Hales–Jewett Theorem answers this: for any number of players, there exists a finite game board dimension such that every fully filled board must contain a winning line.

Words, Variable Words, and Combinatorial Lines

Let $[k] = \{1, 2, \dots, k\}$ be an alphabet of k symbols. **Words** of length n are sequences

$$[k]^n = \{(x_1, \dots, x_n) : x_i \in [k]\}.$$

We view $[k]^n$ as a k -ary n -dimensional discrete cube.

A **variable word** is a word with at least one “wildcard” symbol \star :

$$w = (w_1, \dots, w_n) \in ([k] \cup \{\star\})^n$$

Let $S(w)$ denote the positions of \star .

Words, Variable Words, and Combinatorial Lines

Substituting a letter $a \in [k]$ into all variable positions gives

$$w(a) = (u_1, \dots, u_n), \quad u_i = \begin{cases} a & \text{if } i \in S(w), \\ w_i & \text{if } i \notin S(w). \end{cases}$$

The **combinatorial line** generated by w is

$$L(w) = \{w(a) : a \in [k]\} \subseteq [k]^n.$$

Example: Constructing a Combinatorial Line

Consider the alphabet $[3] = \{1, 2, 3\}$ and words of length $n = 3$.

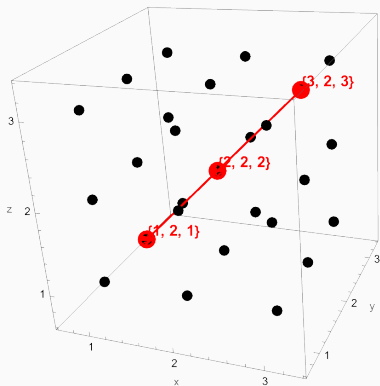
Take the variable word

$$w = (\star, 2, \star).$$

Substituting each $a \in [3]$ into the variable positions gives the combinatorial line

$$L(w) = \{(1, 2, 1), (2, 2, 2), (3, 2, 3)\} \subseteq [3]^3.$$

Example: Visualizing a Combinatorial Line



The combinatorial line $\{(1, 2, 1), (2, 2, 2), (3, 2, 3)\} \subseteq [3]^3$.

The Hales–Jewett Theorem and Hales–Jewett Numbers

Theorem (The Hales–Jewett Theorem [1])

For every $k, r \in \mathbb{Z}^+$, there exists a positive integer $HJ(k; r)$ such that for all $n \geq HJ(k; r)$, every r -coloring of the points in the discrete cube $[k]^n$ admits a monochromatic combinatorial line.

The Hales–Jewett Theorem motivates the following numerical invariant.

Definition (The Hales–Jewett Numbers)

For $k, r \geq 1$, the *Hales–Jewett number* $HJ(k; r)$ is the least integer n such that every r -coloring of $[k]^n$ admits a monochromatic combinatorial line.

In this work, we are concerned with unknown Hales–Jewett numbers

$$HJ(4; 2) \quad \text{and} \quad HJ(3; 3).$$

Best-known Upper Bounds

$$HJ(4; 2) \leq 10^{11}, \quad HJ(3; 3) \leq 2^{2^{3c}},$$

where $c > 0$ is an absolute constant.

Hales–Jewett Lower Bounds via the van der Waerden Numbers

The Hales–Jewett Theorem is a strict generalization of van der Waerden's Theorem.

Theorem (Van der Waerden's Theorem [2])

For integers $k, r \in \mathbb{Z}_{>0}$, there exists an integer $w(k; r)$ such that for all $N \geq w(k; r)$, every r -coloring of $\{1, 2, \dots, N\}$ contains a monochromatic k -term arithmetic progression.

The minimal such integer $w(k; r)$ is called the *van der Waerden number*.

Every combinatorial line in $[k]^n$ can be mapped to a k -term arithmetic progression in the integers, proving van der Waerden's theorem.

Parameters and Assumption

Let $HJ(k; r) = n$ and define the map

$$f_{k,n} : [k]^n \rightarrow \mathbb{Z}^+, \quad f_{k,n}(\mathbf{x}) := 1 + \sum_{i=1}^n (x_i - 1).$$

- **Interpretation:** maps a point in $[k]^n$ to an integer.
- **Purpose:** translates combinatorial lines into arithmetic progressions.

Lines to Progressions

Let w be a variable word over $[k]^n$ with combinatorial line

$$L(w) = \{w(1), w(2), \dots, w(k)\}, \quad |S(w)| = t.$$

Observation: For all $a \in \{1, \dots, k-1\}$,

$$f_{k,n}(w(a+1)) - f_{k,n}(w(a)) = t.$$

Conclusion: The sequence

$$(f_{k,n}(w(1)), f_{k,n}(w(2)), \dots, f_{k,n}(w(k)))$$

forms a k -term arithmetic progression with common difference t .

Monochromatic Lines Force Monochromatic Progressions

Hales–Jewett guarantee: Every r -coloring of $[k]^n$ contains a monochromatic combinatorial line.

Mapping: Applying $f_{k,n}$ sends this line to a monochromatic k -term arithmetic progression.

Result: This proves van der Waerden's theorem.

Lower Bound Set Up

Let $k, r, n, N \in \mathbb{Z}^+$ satisfy

$$1 + n(k - 1) \leq N.$$

Assume there exists an r -coloring of $[N]$,

$$C_{VW,N} : [N] \rightarrow \mathcal{C}_r = \{c_1, c_2, \dots, c_r\},$$

that *avoids monochromatic k -term arithmetic progressions*.

Definition. Define the induced coloring

$$C_{HJ(k;r),n} : [k]^n \rightarrow \mathcal{C}_r$$

by

$$C_{HJ(k;r),n}(\mathbf{x}) := C_{VW,N}(f_{k,n}(\mathbf{x})).$$

Well-definedness. For all $\mathbf{x} \in [k]^n$,

$$f_{k,n}(\mathbf{x}) \in \{1, 2, \dots, 1 + n(k - 1)\} \subseteq [N],$$

so the definition is valid.

Why the Induced Coloring Has No Monochromatic Lines

Recall that each combinatorial line maps to an arithmetic progression in $[N]$.

Key Observation

Because $C_{VW,N}$ contains no monochromatic k -term arithmetic progressions, the colors assigned by $C_{HJ(k;r),n}$ to the points on a combinatorial line cannot all be the same.

Conclusion. The Hales–Jewett coloring $C_{HJ(k;r),n}$ admits no monochromatic combinatorial line, and $HJ(k; r) \geq n + 1$.

Proposition (Lower bound via van der Waerden numbers)

For $k, r \in \mathbb{Z}^+$ we have

$$HJ(k; r) \geq \frac{w(k; r) - 1}{k - 1}.$$

Consequences. Using the known values

$$w(4; 2) = 35 \quad \text{and} \quad w(3; 3) = 26,$$

we obtain the lower bounds

$$HJ(4; 2) \geq 12 \quad \text{and} \quad HJ(3; 3) \geq 13.$$

Case $HJ(4; 2) \geq 12$:

- Cube colorings: $2^{4^{12}} = 2^{16,777,216}$
- Combinatorial lines: 227,363,409

Case $HJ(3; 3) \geq 13$:

- Cube colorings: $3^{3^{13}} = 3^{1,594,323} \approx 2^{2,526,942}$
- Combinatorial lines: 65,514,541

Boolean Satisfiability

A *Boolean function* is a map

$$f : \{0, 1\}^n \rightarrow \{0, 1\}.$$

Boolean functions are built using the logical connectives conjunction (\wedge), disjunction (\vee), and negation (\neg).

Conjunction plays a central role: a formula

$$\Phi = \Phi_1 \wedge \Phi_2 \wedge \cdots \wedge \Phi_m$$

is true if and only if every subformula Φ_i is true.

Conjunctive Normal Form

A formula is in *conjunctive normal form* (CNF) if it is a conjunction of disjunctions.

The following is a CNF formula with two clauses:

$$(b_1 \vee \neg b_2) \wedge (\neg b_3 \vee b_4 \vee \neg b_5)$$

The Boolean Satisfiability Problem (SAT)

The *Boolean satisfiability problem* (SAT) asks whether a CNF formula Φ has an assignment

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \{0, 1\}^n$$

such that

$$\Phi(\alpha_1, \dots, \alpha_n) = 1.$$

If such an assignment exists, Φ is *satisfiable*; otherwise it is *unsatisfiable*.

- The SAT problem is NP-complete, but modern SAT solvers can efficiently handle CNF formulas with millions of variables and clauses.
- We solve our SAT instances using **CryptoMiniSat**, a state-of-the-art parallel SAT solver.

Hales–Jewett and SAT

Goal. For fixed $k, r, n \in \mathbb{Z}^+$, determine whether there exists a coloring

$$C_{HJ(k;r),n} : [k]^n \rightarrow \mathcal{C}_r$$

with no monochromatic combinatorial line.

SAT encoding. We encode this existence problem as a Boolean formula $\Phi_{[k]^n}$ such that

- If $\Phi_{[k]^n}$ is satisfiable, then $HJ(k; r) \geq n + 1$.
- If $\Phi_{[k]^n}$ is unsatisfiable, then $HJ(k; r) \leq n$.

Encoding $HJ(4; 2)$

Each point $\mathbf{x} \in [4]^n$ is assigned a Boolean variable $b_{j_{\mathbf{x}}}$:

$b_{j_{\mathbf{x}}} = 1 \iff \mathbf{x}$ is colored red, $b_{j_{\mathbf{x}}} = 0 \iff \mathbf{x}$ is colored blue,

where

$$j_{\mathbf{x}} = 1 + \sum_{i=1}^n 4^{n-i}(x_i - 1).$$

Encoding $HJ(4; 2)$

To forbid combinatorial line

$$L(w) = \{w(1), w(2), w(3), w(4)\} \subseteq [4]^n,$$

from being monochromatic, we add clauses

$$(\neg b_{j_{w(1)}} \vee \neg b_{j_{w(2)}} \vee \neg b_{j_{w(3)}} \vee \neg b_{j_{w(4)}}) \quad \text{and} \quad (b_{j_{w(1)}} \vee b_{j_{w(2)}} \vee b_{j_{w(3)}} \vee b_{j_{w(4)}}).$$

Let \mathcal{L} denote the set of combinatorial lines in $[4]^n$. The full CNF formula is

$$\Phi_{[4]^n} = \bigwedge_{L \in \mathcal{L}} \left[(\neg b_{j_w(1)} \vee \neg b_{j_w(2)} \vee \neg b_{j_w(3)} \vee \neg b_{j_w(4)}) \right. \\ \left. \wedge (b_{j_w(1)} \vee b_{j_w(2)} \vee b_{j_w(3)} \vee b_{j_w(4)}) \right].$$

Interpretation. $\Phi_{[4]^n}$ is satisfiable iff there exists a 2-coloring of $[4]^n$ with no monochromatic combinatorial line.

Encoding $HJ(3; 3)$

In the 3-color case, each point $\mathbf{x} \in [3]^n$ is represented by three Boolean variables $(b_{j_{\mathbf{x}},r}^{\tilde{j}_{\mathbf{x}}}, b_{j_{\mathbf{x}},b}^{\tilde{j}_{\mathbf{x}}}, b_{j_{\mathbf{x}},g}^{\tilde{j}_{\mathbf{x}}})$:

$$b_{j_{\mathbf{x}},r}^{\tilde{j}_{\mathbf{x}}} = \begin{cases} 1, & \text{if } \mathbf{x} \text{ is red,} \\ 0, & \text{otherwise,} \end{cases}$$

$$b_{j_{\mathbf{x}},b}^{\tilde{j}_{\mathbf{x}}} = \begin{cases} 1, & \text{if } \mathbf{x} \text{ is blue,} \\ 0, & \text{otherwise,} \end{cases} \quad b_{j_{\mathbf{x}},g}^{\tilde{j}_{\mathbf{x}}} = \begin{cases} 1, & \text{if } \mathbf{x} \text{ is green,} \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\tilde{j}_{\mathbf{x}} = 1 + \sum_{i=1}^n 3^{n-i}(x_i - 1).$$

To forbid combinatorial line

$$L(w) = \{w(1), w(2), w(3)\} \subseteq [3]^n$$

from being monochromatic, we add the clauses

$$\begin{aligned} & (\neg b_{j_{w(1)},r}^{\sim} \vee \neg b_{j_{w(2)},r}^{\sim} \vee \neg b_{j_{w(3)},r}^{\sim}), \\ & (\neg b_{j_{w(1)},b}^{\sim} \vee \neg b_{j_{w(2)},b}^{\sim} \vee \neg b_{j_{w(3)},b}^{\sim}), \quad (\neg b_{j_{w(1)},g}^{\sim} \vee \neg b_{j_{w(2)},g}^{\sim} \vee \neg b_{j_{w(3)},g}^{\sim}). \end{aligned}$$

At least one color per point. For each $a \in \{1, 2, 3\}$, we add the clause

$$(b_{J_w(a),r}^{\sim} \vee b_{J_w(a),b}^{\sim} \vee b_{J_w(a),g}^{\sim}).$$

A point may be assigned multiple colors, yielding multiple valid colorings.

Computational Verification of Hales–Jewett Lower Bounds

Goal. Obtain explicit colorings that prove bounds

$$HJ(4; 2) \geq 12 \quad \text{and} \quad HJ(3; 3) \geq 13.$$

Using the known values

$$w(4; 2) = 35 \quad \text{and} \quad w(3; 3) = 27,$$

we fix explicit colorings of $[34]$ and $[26]$ that avoid monochromatic arithmetic progressions of lengths 4 and 3, respectively.

Lifting into the Hales–Jewett Cube

Using the map $f_{k,n}$, we lift these van der Waerden colorings into Hales–Jewett cubes.

Binary case. For $\mathbf{x} \in [4]^{11}$, define

$$C_{HJ(4;2),11}(\mathbf{x}) := C_{VW,[34]}(f_{4,11}(\mathbf{x})).$$

Ternary case. For $\mathbf{x} \in [3]^{12}$, define

$$C_{HJ(3;3),12}(\mathbf{x}) := C_{VW,[26]}(f_{3,12}(\mathbf{x})).$$

These colorings assign values to the SAT variables.

Given these assignments, the SAT solver provides explicit colorings that prove

$$HJ(4; 2) \geq 12, \quad HJ(3; 3) \geq 13.$$

Conclusion. These colorings serve as machine-assisted proofs of the stated lower bounds.

An Improved Hales–Jewett Bound

Extending to a Partial 3-Coloring of $[3]^{13}$

Construction.

For $\mathbf{x} \in [3]^{13}$, if $f_{3,13}(\mathbf{x})$ lies in

$$\{1, \dots, 3^{12}\},$$

then assign to \mathbf{x} the color

$$C_{HJ(3;3),13} := C_{HJ(3;3),12}(f_{3,13}(\mathbf{x})).$$

Interpretation.

This construction colors exactly 3^{12} of the 3^{13} points in $[3]^{13}$, i.e. one 12-dimensional slice (one third of the cube).

Verifying $HJ(3; 3) \geq 14$

We identified 9 explicit 3-colorings of $[3]^{13}$ with no monochromatic lines. Hence, we establish the improved lower bound:

$$HJ(3; 3) \geq 14.$$

Novelty. Previous claims rely on the assertion $w(3; 3) > 27$, which is false. Our computation provides the first rigorous proof of this bound.

Off-diagonal Hales–Jewett Numbers

Definition (Partial combinatorial line)

Let $n, k \in \mathbb{Z}_{>0}$, and let $L(w) = \{w(a) : a \in [k]\}$ be a (full) combinatorial line in $[k]^n$. A *partial combinatorial line of length m* is any subset of the form

$$\{w(a) : a \in \{i + 1, i + 2, \dots, i + m\}\},$$

for $0 \leq i \leq k - m$.

Definition (2-colored off-diagonal Hales–Jewett numbers)

Let $k, m \in \mathbb{Z}^+$ with $m \leq k$.

The *2-color off-diagonal Hales–Jewett number* $HJ(k, m; 2)$ is the smallest $n \in \mathbb{Z}^+$ such that every 2-coloring

$$\chi : [k]^n \rightarrow \{\text{red, blue}\}$$

admits at least one of the following configurations:

- a red monochromatic combinatorial line of length k ;
- a blue monochromatic partial combinatorial line of length m .

3-colored Off-diagonal Hales–Jewett Numbers

Definition (3-colored off-diagonal Hales–Jewett numbers)

Let $j, m, k \in \mathbb{Z}^+$ with $j \leq m \leq k$.

The *3-color off-diagonal Hales–Jewett number* $HJ(k, m, j; 3)$ is the smallest $n \in \mathbb{Z}^+$ such that every 3-coloring

$$\chi : [k]^n \rightarrow \{\text{red, blue, green}\}$$

admits at least one of the following configurations:

- a red monochromatic combinatorial line of length k ;
- a blue monochromatic partial combinatorial line of length m ;
- a green monochromatic partial combinatorial line of length j .

2-colored Off-Diagonal Results

Using our SAT-based encoding, we compute the exact values

$$HJ(3, 2; 2) = 3, \quad HJ(4, 2; 2) = 4,$$

$$HJ(5, 2; 2) = 5, \quad HJ(6, 2; 2) = 6.$$

These computations suggest the following conjecture.

Conjecture ($HJ(k, 2; 2) = k$)

Let $k \in \mathbb{Z}^+$ with $k \geq 2$. Then

$$HJ(k, 2; 2) = k.$$

Partial progress. We have shown that

$$HJ(4, 3; 2) \geq 8.$$

3-colored Off-Diagonal Results

We have proved that

$$HJ(3, 2, 2; 3) = 5.$$

Partial progress.

We have also established that

$$HJ(3, 3, 2; 3) \geq 9.$$

Future Directions

Toward $HJ(k, 2; 2) = k$

Conjecture

For all $k \in \mathbb{Z}^+$ with $k \geq 2$,

$$HJ(k, 2; 2) = k.$$

Current Results

U.B. Theorem (Conlon)

$$HJ(k, 2; 2) \leq k.$$

L.B. Theorem (Conlon, Farnsworth, Robertson, in prep.)

If t is the smallest integer such that $k - t$ is prime, then

$$HJ(k, 2; 2) \geq k - t.$$

Corollary (Conlon, Farnsworth, Robertson, in prep.)

If p is prime, then

$$HJ(p, 2; 2) = p.$$

To prove the conjecture, it suffices to show

$$HJ(k, 2; 2) \geq k \quad \text{for all composite } k \in \mathbb{Z}^+.$$

Additional extensions of this work include:

- Applying the SAT-based framework to geometric Hales–Jewett numbers.
- Developing improved SAT-based algorithms tailored to Hales–Jewett instances.

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References

- [1] Aaron Robertson. *Fundamentals of Ramsey Theory*. CRC Press (2021). DOI: 10.1201/9780429431418.
- [2] B. L. van der Waerden. *Beweis einer Baudetschen Vermutung*. Nieuw Archief voor Wiskunde. Tweede Serie, 15:212–216 (1927).

Appendix A: Showing $HJ(p, 2; 2) \geq p$

Consider a 2-coloring of $[p]^{p-1}$ and color a point \mathbf{x} **blue** iff

$$\sum_{i=1}^{p-1} x_i \equiv 0 \pmod{p}.$$

Along a line the sum becomes

$$a + td \pmod{p},$$

and we obtain a blue point exactly when $a + td \equiv 0 \pmod{p}$.

Since p is prime, d is invertible mod p , so a solution

$$t \equiv -ad^{-1} \pmod{p}$$

always exists.