Bootstrap Percolation and Partial Difference Equations

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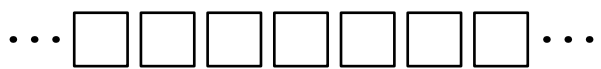
Part 1

The Line

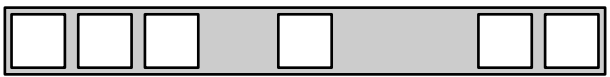
Game

Strip of cells

Paint green



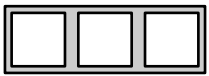
Stencil



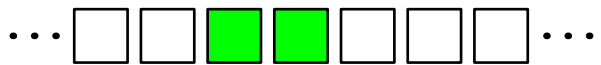
If all but one are green,
we can complete

Must begin somewhere
Seed

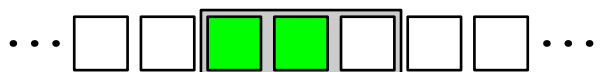
Example
Stencil



Seed



Step



Order of moves doesn't matter
May view as process
"Bootstrap percolation"

Problem
Smallest seed?
(Given stencil)

How to solve?

Fibonacci sequence

. . ., -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, . . .

Any initial values

Secret, want to reveal
"Paint green" $\Rightarrow$ "Uncover term"
Fewest hints?

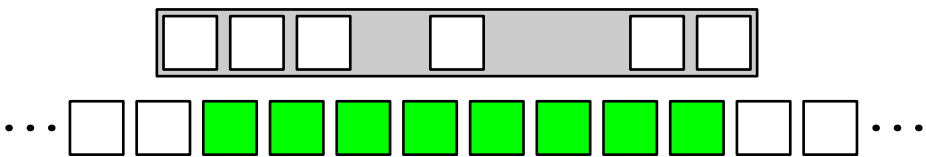
How many Fibonacci sequences?
Two-dimensional vector space
$\Rightarrow$ Two hints

In general
Fix linear recurrence
Vector space of sequences
Dimension?

Theorem

On the line, the smallest seed
is one smaller than
the stencil's enveloping window

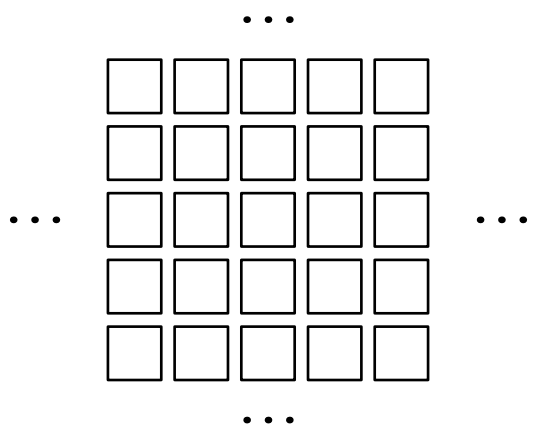
Example



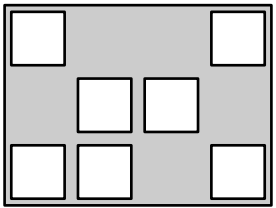
Part 2

The Plane

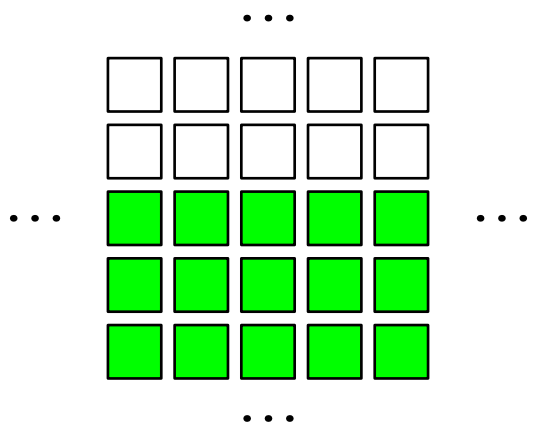
What about two dimensions?



Same rules
Two-dimensional stencil

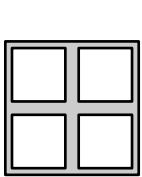


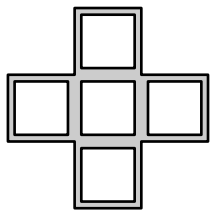
Half-space trap



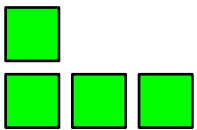
Single stencil
No finite seed

Pair of stencils
Example

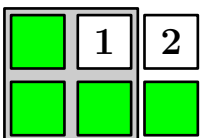




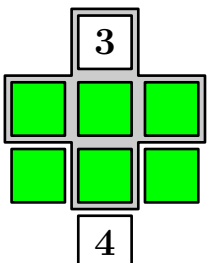
Seed



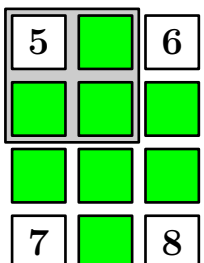
Step



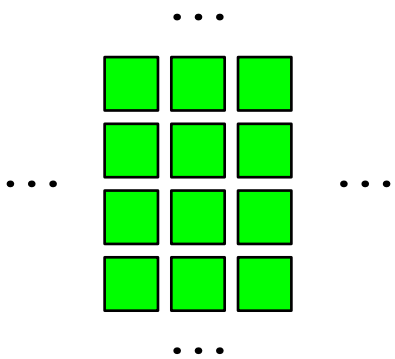
Step



Step



Step



In general?

On the line
Vector space of sequences

In the plane
Vector space of arrays

Example
Every square sums to zero
Every cross sums to zero

Dimension?
Special seed

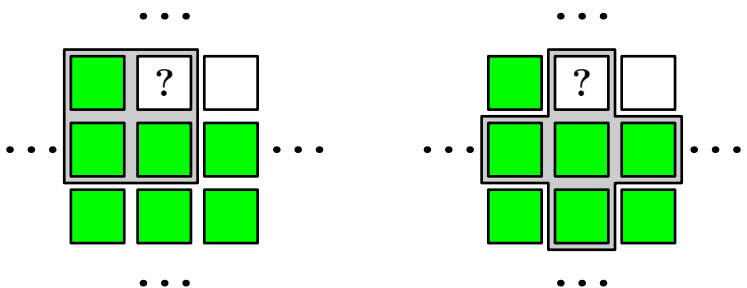
On the line
Consecutive block

In the plane
Tricky construction

On the line
Compute directly

In the plane
What if overlap?

Some values forced twice



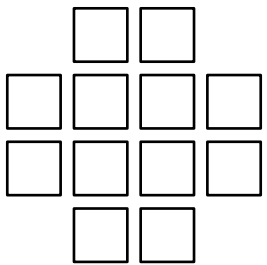
What if contradiction?

Can we avoid overlaps?

No, we cannot
Minkowski sum
$A + B = \{a + b \mid a \in A \text{ and } b \in B\}$

Example

Square and cross



Each cell in both

Last cell forced twice

There will be overlap
But contradiction?

Lemma

Minkowski sum all fine

Lemma

We can set up our process so that this is the only issue that arises

$\Rightarrow$ Overlaps always like this

$\Rightarrow$ Agreement everywhere

$\Rightarrow$ Any initial values yield array

$\Rightarrow$ Dimension

Same overall strategy
But with tricky overlaps

Algebraic reasoning
Fine in Minkowski sum

Combinatorial reasoning
Hence, fine everywhere

Theorem

In the plane, the smallest seed size
equals the mixed volume
of the stencils' convex hulls

What about higher dimensions?

Part 3

Equations

Think back to the plane
The combinatorial part
doesn't care about the algebraic part

The algebraic slot
could contain any polynomials
The combinatorics still works

“Partial difference equations”
Consistency

Theorem

Consistent on Minkowski sum

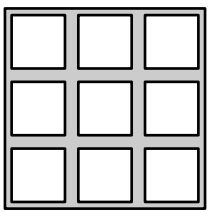
$\Leftrightarrow$

Consistent everywhere

True of all linear
How about non-linear?

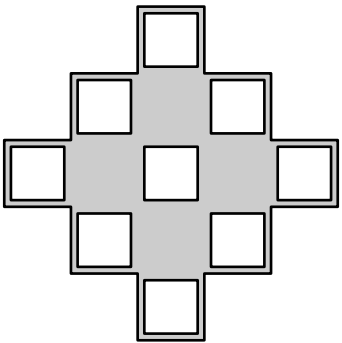
Example
Determinants

Square



Determinant $= 0$

Diamond



Determinant $= 0$

$SD(m, n)$

Square $m \times m$

Diamond $n \times n$

Consistent?

$SD(2, 2)$, $SD(2, 3)$, $SD(3, 2)$

Applications
$SD(2,3)$
Somos sequences

Recurrence

$$s_{i+4} = \frac{a_1 s_{i+1} s_{i+3} + a_2 s_{i+2}^2}{s_i}$$

Multiply by itself

$$a_{i,j} = s_i s_j$$

Determinant identities
Decimation

What about other systems?
Experiments

What about consistency criteria
in higher dimensions?

Thank you!