

Resolving the Hypercube

Victor S. Miller

SRI & Anduril Industries

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Experimental Mathematics Seminar

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A discrete version of GPS

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Resolving Set

Set of points: distances from them uniquely determine an object.

Metric Dimension

Smallest size of a resolving set [4].

$\dim(\mathbb{R}^n) = n + 1$ (affine dimension).

Generalize to *metric spaces*.

Applications

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- Coin weighing [5]
- Fingerprinting for digital data.
- Network discovery and verification.
- Multiuser coding and CDMA [15]
- Compressed Genotyping [9].
- Group Testing [10, 1]
- Robot Navigation.
- Drug Discovery.
- Mastermind and Wordle.

Graphs

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- G : connected, simple, undirected graph.
- $d_G(u, v)$: length of shortest path from u to v in G .
- $u \in V(G)$ *resolves* $v \neq w \in V(G)$ if $d_G(u, v) \neq d_G(u, w)$.
- $S \subseteq V(G)$ *resolving set*: all pairs $v \neq w$ are resolved by some $u \in S$.
- NP-hard for general graphs ☹.

Hitting Sets

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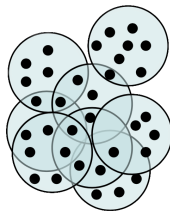
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- Universe: $U, S_1, \dots, S_m \subseteq U$.
- Hitting Set: $H \subset U, H \cap S_i \neq \emptyset$.
- Find a hitting set of *minimum size*.
- Also NP-complete ☹.
- Metric Dimension: $U = V(G)$,
 $S_{\{u,v\}} = \{x : d_G(x, u) \neq d_G(x, v)\}$.
- Straightforward formulation for Integer Programming or MAXSAT.
- Finding small resolving sets is fairly easy. Proving minimality is hard.



Erdős-Rényi 1963

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ON TWO PROBLEMS OF INFORMATION THEORY

by

PAUL ERDŐS and ALFRÉD RÉNYI

§ 1. Introduction

The first problem¹ which will be discussed in this paper can be formulated as follows: Suppose we are given n coins, which look quite alike, but of which some are false. (For instance suppose that the right coins consist of gold, while the false coins consist mainly of silver and are covered only by a thin layer of gold.) The false coins have a smaller weight than the right coins; the weights a and $b < a$ of both the right and false coins are known. A scale is given by means of which any number $\leq n$ of coins can be weighed together. Thus if we select an arbitrary subset of the coins and put them together on the scale, then the scale shows us the total weight of these coins, from which it is easy to compute the number of false coins among those weighed. The question is what is the minimal number $A(n)$ of weighings by means of which the right and false coins can be separated? It can be seen by an elementary information-theore-

Finally let us mention the following geometric interpretation of both problems. To any sequence of zeros and ones there corresponds a vertex of the unit cube C_n of n -dimensional space. The function $B(n)$ can be interpreted as follows: $B(n)$ denotes the least number such that by selecting $B(n)$ suitable chosen vertices of C_n each vertex of C_n is uniquely determined by its distances from the chosen $B(n)$ vertices.

The hypercube

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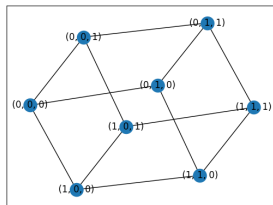
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- Graph Q^n .
- Nodes: n -tuples of 0/1.
- Edges: differ in 1 coordinate.
- Hamming Weight:
 $w(u)$: Number of 1's.
- $d_{Q^n}(u, v) = w(u \oplus v)$
- β_n : Metric Dimension of Q^n (A303735).
- Erdős-Rényi [8]
Problem B.



Coin Weighing

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- n coins, some are counterfeit smaller weight.
- γ_n : smallest number of non adaptive weighings for n coins to find counterfeit.
- $\beta_n - 1 \leq \gamma_n \leq \beta_n$
- Erdős-Rényi Problem A.



Coin Weighing History

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- Introduced by Söderberg-Shapiro [17] in 1963.
- Erdős-Rényi [8] problem A.
- Constructions by Lindström [13] and Cantor & Mills [6] close to optimal value.
- Asymptotics $\gamma_n \sim \beta_n \sim \frac{2n}{\log_2 n}$.

Matrices

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Matrix model

- A : $m \times n$ matrix with entries in $\{0, 1\}$.
- $T(n)$: nonzero n vector with entries in $\{0, \pm 1\}$.
- $B(n) = \{x \in T(n) : \sum_i x_i = 0\}$.
- A is *detecting*: $\forall x \in T(n) : Ax \neq 0$.
- A is *balanced detecting*: $\forall x \in B(n) : Ax \neq 0$.

Proposition

We have $\gamma_n \geq m$ if and only if there is an $m \times n$ detecting matrix.

Proposition

We have $\beta_n \geq m$ if and only if there is an $(m - 1) \times n$ balanced detecting matrix.

Simple bounds

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- $\beta_n \leq \beta_{n+1}$: Drop last column of a balanced detecting matrix for $n + 1$.
- $\beta_{n+1} \leq \beta_n + 1$: Let v be any row of A , an $\beta_n \times n$ balanced detecting matrix.

$$A' = \left[\begin{array}{c|c} v & 1 \\ \hline \mathbf{A} & 0 \end{array} \right]$$

Bounds for γ_n

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Lower bound: from Pippenger's analysis [16] using entropy.
Upper bounds: from various papers using search heuristics [14].
Lindström and Cantor-Mills have explicit constructions for detecting matrices.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
lower	1	2	3	3	4	4	4	4	5	5	6	6	6	6	7	7	7	7	8	8	8	8
upper	1	2	3	3	4	5	6	6	7	8	9	7	8	9	10	10	11	12	13	13	14	15

Pippenger's argument

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$H(X) = - \sum_j \Pr(X = j) \log_2 \Pr(X = j)$: binary entropy

Proposition (Pippenger)

- A : $m \times n$ $0/1$ detecting matrix.
- D_j : j rows of A as a set.
- $X \subset [n]$ random, $Y^{(D)} := \#(D \cap X)$.
- $\Pr(U^{(d)} = j) = \binom{d}{j} 2^{-d}, j = 0, 1, \dots, d$.
- $U^{(\#D)}$ same distribution as $Y^{(D)}$.

$$n = H(Y^{(D_1)}, \dots, Y^{(D_m)}). \quad (1)$$

Corollary

$$m \geq \left\lceil \min_{d=1}^n \frac{n}{H(U^{(d)})} \right\rceil.$$

Balanced Matrices

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Proposition

- A : $m \times n$ 0/1 balanced detecting matrix.
- $D_j \subset [n]$ the set of the j -th row.
- $X^{(s)}$ random $\#X^{(s)} = s$, $Y^{(D,s)} = \#(D \cap X^{(s)})$
- $\Pr(U^{(d,s)} = j) = \frac{\binom{d}{j} \binom{n-d}{s-j}}{\binom{n}{s}}.$

Then for $1 \leq s \leq \lfloor n/2 \rfloor$

$$1 = \frac{H(X^{(D_1,s)}, \dots, X^{(D_m,s)})}{H(X^{(s)})} \leq \sum_{j=1}^m \frac{H(U^{(\#D_j,s)})}{\log_2 \binom{n}{s}}. \quad (2)$$

Corollary

$$m \geq \left\lceil \min_{d=1}^n \frac{\log_2 \binom{n}{s}}{H(U^{(d,s)})} \right\rceil, 1 \leq s \leq \lfloor n/2 \rfloor.$$

Some bounds

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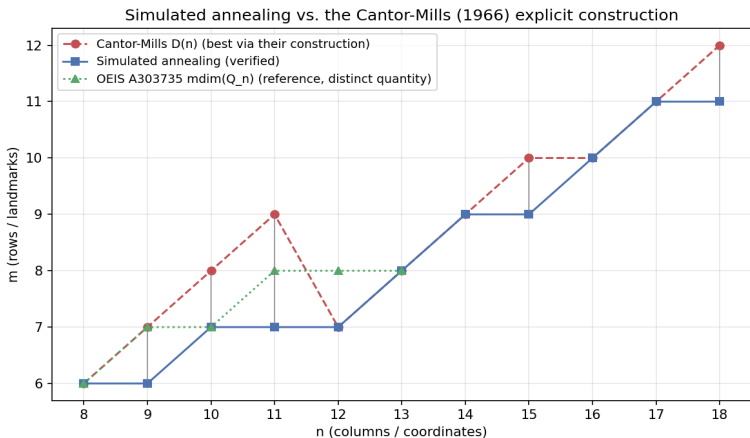
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Simulated Annealing

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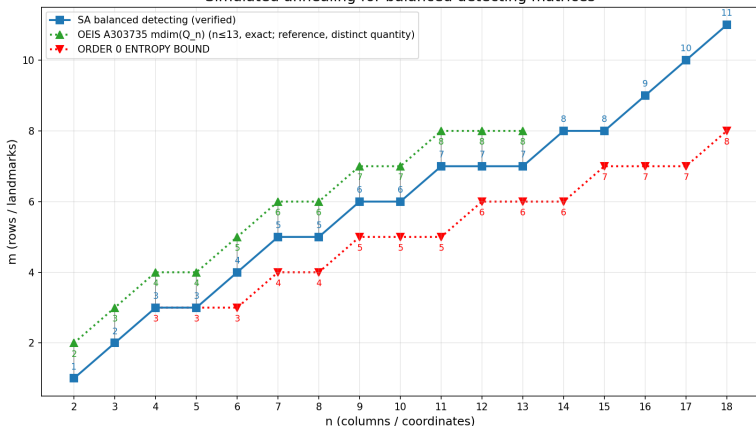
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Simulated annealing for balanced detecting matrices



Symmetry Breaking

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- Many problems: there is a group acting on potential solutions.
- Paths in search tree that are equivalent under group must be explored.
- Solution: Enforce one orbit representative (or a small number).
- Luks: breaking all symmetries is NP-hard.
- Simple case: variables may be permuted: sort them lexicographically.
- Alternative: Generate predicates “on the fly” during search.

Symmetries of Balanced Detecting Matrices

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- Rows of A can be permuted.
- Columns of A can be permuted.
- Complement rows: row weight $\leq n/2$.
- XOR one row into all other rows.
- Group size: $(m+1)!n!2^n$
- Balanced vectors with exactly one ± 1 enforce unequal columns.
- Weight non-decreasing.
- Ties broken with lexicographic order.

Regional Restrictions

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Regions of a weight-sorted 5×10 0/1 matrix

	1	2	3	4	5	6	7	8	9	10	
row 1	1	0	0	0	0	0	0	0	0	0	wt 1, 1 region
row 2	0	1	1	1	0	0	0	0	0	0	wt 3, 2 regions
row 3	0	1	0	0	1	1	1	0	0	0	wt 4, 3 regions
row 4	1	1	0	0	1	0	0	1	0	0	wt 4, 5 regions ← tied, lex > row 3
row 5	1	0	1	0	0	1	0	1	1	0	wt 5, 7 regions

A *region* of row k is a maximal run of adjacent columns whose entries in rows $1, \dots, k-1$ all agree (row 1 is a single region by convention); within every region the row's own 1's precede its 0's. Rows are additionally ordered so weight is non-decreasing top to bottom and never exceeds $n/2 = 5$; where two adjacent rows tie in weight (rows 3, 4 here), the lower one is required to be lexicographically greater. Color is inherited by a region's left-most sub-region when it splits, with every other sub-region drawn in a newly introduced color.

Lazy Constraints

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- $\#B(n) = \frac{A002426(n)-1}{2}$. Grows like $\frac{3^n}{\sqrt{n}}$.
- $A = m \times n$ 0/1 matrices.
- $X = 0/1$ n vectors.
- Formulation: $\forall a \in A, f(a), \exists x \in X, g(x) \Rightarrow \phi(a, x)$.
- a : variables describing balanced detecting matrices, $f(a)$ condition on a .
- x : variables describing vectors, $g(x)$ condition on x .
- $\phi(a, x)$: predicate stating that a detects x .

Ping Pong

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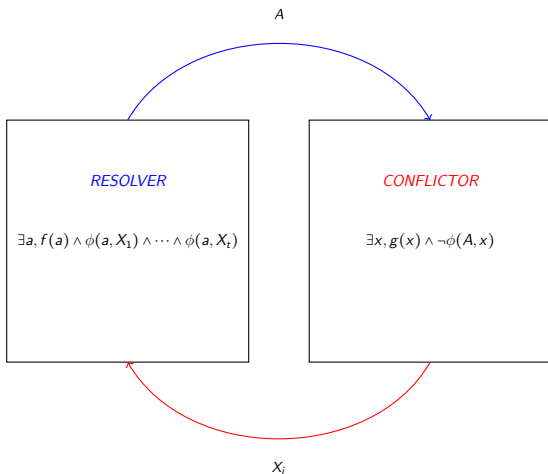
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Restricting rows

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- At critical m find all multisets (d_j) so that

$$1 \leq \sum_{j=1}^m \frac{H(U^{(d_j, s)})}{\log_2 \binom{n}{s}}, s = 1, \dots, \lfloor n/2 \rfloor.$$

Number of multisets of row weights

dimension	3	4	5	6	7	8	9	10	11	12	13	14
original	1	3	3	10	15	35	56	126	210	462	462	1716
reduced	0	0	0	1	8	9	35	49	144	237	94	1059

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- Performed on Apple M3.
- Used Python 3.13 with `python-sat` [12] package and solver `Cadical` [3].
- Heavily dependent on good cardinality encoding.
- Conflictor generates up to 10 counterexamples to give to Resolver.
- Confirms that $\beta_{10} = 7$ in under 3.5 seconds.
- Confirms that $\beta_{11} = 8$ in 2300 seconds.
- Along with $\beta_{13} \leq 8$ implies $\beta_{13} = 8$.
- Only needs to generate a small fraction of all x vectors to show UNSAT.

Speed ups?

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- Better choice of counterexamples.
- Use of Shearer's lemma [7] for entropy to cut down on cases.
- Only a small number of vectors in $B(n)$ are necessary.
- Set cover: if $v \in B(n)$, $C(v) = \{A : Av = 0\}$.
- For example: $n = 10$ only 69 out of 4476 vectors needed.

Certifying Proofs

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- Why trust a long calculation that says UNSAT?
- Verify the model.
- Many SAT solvers can emit a DRUP proof of UNSAT.
- Usually long, but easily checked by small program.
- For UNSAT, some SMT solvers can produce a *Skolem function*: $f(A)$ from matrices to vectors such that $Af(A) = 0$ for all A .
- Is A balanced detecting? Lattice Reduction.
- Farkas certificates.

Late News

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Recently Shmulik Cohen contacted me saying that using the orderly isomorph rejection techniques of Faradzen-Read he's independently verified my values and extended the calculation to $n = 14$.

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