A criterion for asymptotic sharpness in the enumeration of simply generated trees

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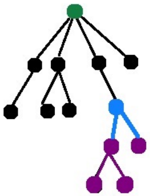
Overview

- Simply generated trees and the main inequality: $y(r) \leq R$
- Criterion for asymptotic sharpness: y(r)=R
- Application in Lie theory: Kuperberg's 1996 conjecture

trees

A **rooted tree** is an undirected acyclic connected graph with a distinguished point, the root. Subtrees dangling from a node are ordered amongst themselves.



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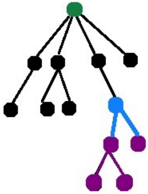

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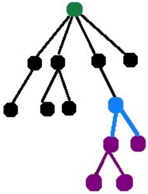

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Example 1: Suppose each node in a tree is allowed to have an arbitrary (non-negative integer) number of children. Such a tree is sometimes called a "planted plane tree." What can be said about y_n, the number of planted plane trees with n nodes?

generating functions

One approach is to use a **generating function**. We let

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In short,

$$\boxed{y(x) = xA(y(x))}$$

where $A(x) = \frac{1}{1-x}$.

generating functions

Idea: A tree of n nodes is built recursively by gluing k subtrees together at the root, for some $1 \leq k < n$. The coefficient $[x^n](xy(x)^k)$ is the number of trees having n nodes and built by gluing k subtrees at the root.

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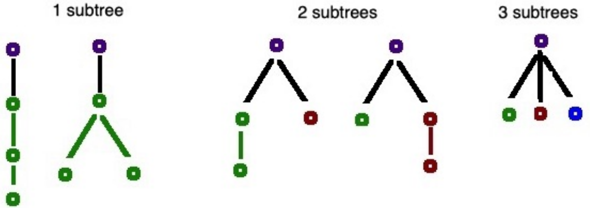
$$\begin{aligned} y_4 &= [x^4]y(x) \\ &= [x^3]y(x) + [x^3]y(x)^2 + [x^3]y(x)^3 = 5 \end{aligned}$$



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The functional equation $y(x) = \frac{x}{1-y(x)}$ can be solved exactly:

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y(x) &= \frac{1-\sqrt{1-4x}}{2} \\
&= x + x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 \cdots
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In fact, $y_{n+1} = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number.

We see immediately that $\limsup_{n\to\infty} \sqrt[n]{y_n} = 4$, by the principle that the radius of convergence of an analytic function is the distance from the origin to its nearest singularity. This gives the **exponential growth**: $y_n \sim 4^{n+o(n)}$, as $n \to \infty$.

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Theorem

For $\lambda \in \mathbb{R}^+$, *and* $\alpha \in \mathbb{C} \setminus \mathbb{Z}^{\geq 0}$, *we have* $[z^n](1 - \lambda z)^\alpha \sim \frac{\lambda^n}{n^{\alpha+1}\Gamma(-\alpha)}$, *as* $n \to \infty$.

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In the case $y(x) = \frac{1-\sqrt{1-4x}}{2}$, this gives (since $\Gamma(-1/2) = -2\sqrt{\pi}$):

$$y_n \sim \frac{-1}{2} \cdot \frac{4^n}{n^{3/2}\Gamma(-1/2)} = \frac{4^{n-1}}{n^{3/2}\sqrt{\pi}} \quad .$$

transfer theorem

This example is nice, but what if we can't solve for $y(x)$ exactly? Or what if we can solve for $y(x)$, but it's not of the form $(1-\lambda z)^\alpha$?

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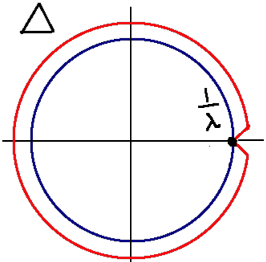
Theorem (Flajolet, Odlyzko, 1990)

For $\lambda \in \mathbb{R}^+$ and $\alpha \in \mathbb{C} \setminus \mathbb{Z}^{\geq 0}$, if $y(x) = \sum_{n\geq 1} y_n x^n$ is analytic in a Pac-Man domain Δ around its disk of convergence, then

$$y(z) \sim (1-\lambda z)^\alpha \text{ as } z \to (1/\lambda) \text{ in } \Delta \implies [x^n]y(x) \sim \frac{\lambda^n}{n^{\alpha+1}\Gamma(-\alpha)} \text{ as } n \to \infty.$$



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Example 2: Let y_n count the number of trees with n nodes, such that each node has 0,1, or 2 children. Then with $A(x) = 1 + x + x^2$, we have

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&= 1 - \sqrt{3}(1-3x)^{1/2} + \mathcal{O}(1-3x).
\end{aligned}$$

$y(x) - 1 \sim -\sqrt{3}(1-3x)^{1/2}$, so that $y_n \sim \frac{-\sqrt{3}\cdot 3^n}{\Gamma(-1/2)n^{3/2}} = \frac{\sqrt{3}}{2\sqrt{\pi}} \cdot \frac{3^n}{n^{3/2}}$.

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Combinatorial interpretation: y_n counts the number of rooted trees with n nodes (including the root), such that for each $i \geq 1$, each internal node having i children can be colored with one of a_i colors. A family of rooted trees is called **simply generated** if counted by a sequence (y_n) satisfying the above functional equation for some $A(x)$ with non-negative coefficients.

previously known facts

Call a sequence $(a_n)_{n\geq 0}$ of non-negative integers "*good*" if:

1. $a_0 = 1$, $a_n \geq 1$ eventually,
2. (a_n) is not supported on an arithmetic progression in $\mathbb{N}$,
3. $y(x)$ has finite radius of convergence r.

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This inequality is what we want to study – when is it sharp?

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(2) *The coefficient sequence $(y_n)_{n=1}^{\infty}$ satisfies the following asymptotic estimate:*

$$y_n = \frac{C}{r^n n^{\frac{3}{2}}}(1 + \mathcal{O}(n^{-1})),$$

as $n \to \infty$, where $C = \sqrt{\frac{A(\tau)}{2\pi A''(\tau)}}$.

Recall Example 1: $y(x) = xA(y(x))$, where $A(x) = \frac{1}{1-x}$, and $R = 1$.
In this case $\psi(x) = x(1-x)$, and $A(x) - xA'(x)$ vanishes at $1/2$.
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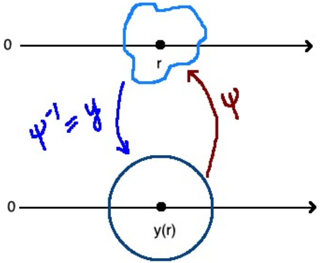
Next, if $y(r) < \tau$, then $\psi'(y(r)) \neq 0$. Therefore, y admits an analytic continuation to a neighborhood of r, namely the local inverse of ψ at r. This contradicts Pringsheim's Theorem, establishing that $y(r) = \tau$.



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We see that

$$y(z) - \tau \sim (z - r)^{1/2} h(r),$$

as $z \to r$, and the transfer theorem of Flajolet/Odlyzko applies. Thus,

$$y_n \sim \frac{C}{r^n n^{3/2}}.$$

criterion for sharpness

Theorem (S., 2020)

Suppose that $(a_n)_{n\geq 0}$ *is good and the generating functions* $A(x)$ *and* $y(x)$*, with radii of convergence* R *and* r *respectively, satisfy* $\boxed{y(x) = xA(y(x))}$.

Then, the inequality $\boxed{y(r) \leq R}$ *is sharp (i.e.* $y(r) = R$*) if* $A(z) - zA'(z)$ *doesn't vanish on* $(0, R)$.

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With $A(x)$ *and* $y(x)$ *as above, exactly one of the following is true:*

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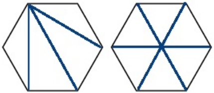
In particular, *the absence of the* $n^{-3/2}$ *polynomial factor in the asymptotic expansion of* y_n *certifies that the inequality* $y(r) \leq R$ *is actually sharp.*

application in Lie theory

For each positive integer n, let a_n denote the number of triangulations of a regular n-gon, such that the minimum degree of each internal vertex is 6. The sequence begins

$$(a_n)_{n=1}^{\infty} = 0, 1, 1, 2, 5, 15, 50, 181, 697, \ldots$$



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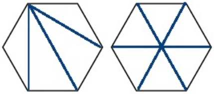

For each positive integer n, let b_n denote the dimension of the space of invariant tensors in the n-th tensor power of the 7-dim fundamental representation of the exceptional simple Lie algebra G_2. The sequence begins

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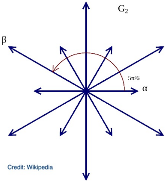

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Theorem (Kuperberg, 1996)

The identity of formal power series holds:

$$\boxed{y(x) = xA(y(x))},$$

Furthermore, $y(x)$ has radius of convergence $r = 1/7$, and hence

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \leq \frac{1}{y(1/7)}.$$

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Conjecture (Kuperberg, 1996)

$$y(r) = R, \qquad i.e. \quad \limsup_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{y(1/7)} \approx 6.811.$$

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Furthermore,

$$b_n = K(7^n/n^7)(1+o(1)), \quad \text{as } n \to \infty,$$

for a constant $K \approx 2627.6$

outline of proof

(1) Derive asymptotics for $(b_n)_{n\geq 1}$:

$$b_n \sim K\frac{7^n}{n^7}, \text{ as } n \to \infty.$$

(2) The asymptotics of $(b_n)_{n\geq 1}$, *specifically the presence of the* n^{-7} *polynomial factor as opposed to* $n^{-3/2}$, indicates by the corollary above that $A(z) - zA'(z)$ does not vanish on $(0, R)$, and hence that $R = y(r) = y(1/7)$.

(3) It follows that

$$\limsup_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{y(1/7)}.$$

With a little more work this becomes an actual limit, implying the asymptotic expression for a_n.

(4) The last step is to evaluate $y(1/7)$ exactly.

some details: (1) asymptotics for $(b_n)_{n\geq 1}$

(1) b_n is the coefficient of $x^n y^n$ in WM^n (Kuperberg, 1994), where

$$M(x,y) = 1 + x + y + xy + x^2y + xy^2 + (xy)^2,$$

and

$$W(x,y) = x^{-2}y^{-3}(x^2y^3 - xy^3 + x^{-1}y^2 - x^{-2}y + x^{-3}y^{-1} - x^{-3}y^{-2} + x^{-2}y^{-3} - x^{-1}y^{-3} + xy^{-2} - x^2y^{-1} + x^3y - x^3y^2).$$

Use a saddle-point analysis to evaluate this coefficient:

$$b_n = \frac{1}{(2\pi i)^2} \oint \oint \left[W(z_1, z_2) \cdot M(z_1, z_2)^n \cdot \frac{1}{(z_1 z_2)^{(n+1)}} \right] dz_1\, dz_2.$$

Surprisingly, integrals involving lower order terms vanish, and we recover the factor n^{-7} as well as

$$K = \frac{4117715\sqrt{3}}{864\pi} \approx 2627.56\,.$$

some details: (4) value of y(1/7)

(4) Proof that $y(1/7) = \frac{5\pi}{8575\pi - 15552\sqrt{3}}$.

Theorem (Bostan, Tirrell, Westbury, Zhang, 2019)

$$y(x) = \frac{1}{30x^4}\left[R_1 \cdot {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 2; \phi(x)\right) + R_2 \cdot {}_2F_1\left(\frac{2}{3}, \frac{4}{3}; 3; \phi(x)\right) + 5P\right],$$

where

$$\begin{aligned} R_1(x) &= (x+1)^2(214x^3 + 45x^2 + 60x + 5)(x-1)^{-1}, \\ R_2(x) &= 6x^2(x+1)^2(101x^2 + 74x + 5)(x-1)^{-2}, \\ \phi(x) &= 27(x+1)x^2(x-1)^{-3}, \\ P(x) &= 28x^4 + 66x^3 + 46x^2 + 15x + 1. \end{aligned}$$

some details: (4) value of y(1/7)

(4) Proof that $y(1/7) = \frac{5\pi}{8575\pi - 15552\sqrt{3}}$.

Evaluating the polynomials at $x = \frac{1}{7}$, the formula simplifies to

$$y\left(\frac{1}{7}\right) = \frac{7^6}{30}\left[\frac{-55296}{2401} \cdot {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 2; 1\right) + \frac{9216}{2401} \cdot {}_2F_1\left(\frac{2}{3}, \frac{4}{3}; 3; 1\right) + \frac{150}{7}\right].$$

We use facts about the gamma function, namely that $\Gamma(z+1) = z\Gamma(z)$ for $z \notin \mathbb{Z}_{\leq 0}$ and the following:

- $${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (\mathrm{Re}(c) > \mathrm{Re}(a+b)),$$
- $$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad (z \in \mathbb{C}).$$

These suffice to simplify the above expression for $y\left(\frac{1}{7}\right)$.

Thank you.

Thank you for the opportunity to speak today!

For more details, there is a draft on the arXiv with the same title as the talk, although a more polished version will be forthcoming soon.