Collatz polynomials: an introduction with bounds on their zeros

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Collatz conjecture

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Define

$$C(N) = \begin{cases} \frac{3N+1}{2} & , N \text{ odd} \\ \frac{N}{2} & , N \text{ even} \end{cases} \tag{1}$$

Then, for all N, there exists n such that

$$C^n(N) = 1$$

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“Mathematics is not yet ready for such problems."

– Paul Erdős

Collatz polynomials: definition

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Definition (N^{th} Collatz polynomial)

The polynomial

$$P_N(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$$

with coefficients

- $a_0 = N$
- $a_k = C^k(N)$
- $a_n = 1$

where n is the *total stopping time* of N, the least n such that $C^n(N) = 1$.

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where n is the *total stopping time* of N, the least n such that $C^n(N) = 1$. (Note that we are assuming Collatz to be true.)

Collatz polynomials: examples

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Example

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$$P_{13}(z) = 13 + 20z + 10z^2 + 5z^3 + 8z^4 + 4z^5 + 2z^6 + z^7$$

Bounds on zeros

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In what follows, let ζ_N be a zero of $P_N(z)$.

Theorem

$$\frac{2N}{3N+1} \le |\zeta_N| \le 2 \qquad (2)$$

Bounds on zeros

Bounds on zeros

Upper bound.

Theorem (Borwein and Erdélyi[a])

[a] Corollary 1.2.3, "Polynomials and Polynomial Inequalities"

If $p(z) = a_n z^n + ... + a_1 z + a_0$ *for* $a_k > 0$ *for all* k*, then all zeros of* $p(z)$ *lie in the annulus*

$$\min_{0 \le k \le n-1} \frac{a_k}{a_{k+1}} \le |z| \le \max_{0 \le k \le n-1} \frac{a_k}{a_{k+1}} \tag{3}$$

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By definition of C,

$$\frac{a_k}{a_{k+1}} = \begin{cases} 2, & a_k \text{ even} \\ \frac{2N}{3N+1}, & a_k \text{ odd} \end{cases}$$

□

Other properties of zeros: real zeros

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For $N \geq 3$, $P_N(z)$ has at least one non-real zero.

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Proof (Part I).

Suppose all zeros are real, and label them in order:

$$-2 \leq r_1 \leq r_2 \leq \cdots \leq r_n \leq 0 \tag{4}$$

where $r_n \leq 0$ by Descartes's Rule of Signs and $-2 \leq r_1$ because

$$-2 = \sum_{i=1}^{n} r_i = -C^{n-1}(N) \tag{5}$$

□

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Proof (Part II).

Suppose $-1 \leq r_1$; then

$$1 \geq \prod_{i=1}^{N} |r_i| = N \geq 3 \tag{6}$$

a contradiction. Thus $-2 \leq r_1 < -1$, implying

$$-1 < r_2 \leq \cdots \leq r_n < 0 \tag{7}$$

Thus

$$1 > \prod_{i=2}^{N} |r_i| = \frac{N}{|r_1|} \geq \frac{3}{2} \tag{8}$$

a contradiction. □

Other properties of zeros: rational/integer zeros

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By the Rational Root Theorem and Descartes's Rule of Signs, $P_N(z)$ has rational root r only if

$$r = -2, -1 \tag{9}$$

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In what follows, let $m(N)$ be the number of odd integers in the trajectory of N.

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Lemma

If N_i is the i^{th} odd number appearing in the Collatz trajectory of N, with each N_i appearing at the end of a subsequence

$$2^{\ell_i - 1} N_i, 2^{\ell_i - 2} N_i, ..., N_i$$

of length ℓ_i, then

$$P_N(z) = \sum_{k=1}^{m(N)} \left(2^{\ell_k} N_k\right) \left(z^{\ell_1 + \ldots + \ell_{k-1}}\right) \left(\frac{1 - \left(\frac{z}{2}\right)^{\ell_k}}{2 - z}\right)$$

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Theorem

With notation as above, $P_N(-2) = 0$ if and only if ℓ_i is even for all $i = 1, ..., m$.

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Proof.

One direction follows by direct substitution of -2 into the formula for $P_N(z)$.
For the other direction, note that the equality

$$0 = P_N(-2) = \sum_{k=1}^{m} \left(2^{\ell_k} N_k\right) \left((-2)^{\ell_1+\ldots+\ell_{k-1}}\right) \left(\frac{1-(-1)^{\ell_k}}{4}\right) \quad (10)$$

$$= \sum_{k:\ \ell_k \text{ odd}} N_k \cdot (-2)^{\ell_1+\ldots+\ell_k-1} \quad (11)$$

is, if at least one ℓ_k is odd, equivalent to -2 being the root of a non-zero polynomial with only odd coefficients. This is impossible, contradiction. □

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Theorem

If

$$P_N(-1) = 0$$

then $m(N)$ is even.

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Proof.

If

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then

$$P_N(1) = 0$$

in $(\mathbb{Z}/2\mathbb{Z})[z]$, but this is true if and only if $P_N(z)$ has an even number of odd coefficients. □

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Theorem

If $N = c^{-1}(2^t)$, where t is the base of N, then $P_N(-1) = 0$.

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Example

$P_5(-1) = P_{21}(-1) = 0$

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Proof.

$$P_N(-1) = \frac{2^{t+1}-1}{3} + 2^{t+1} \cdot \sum_{k=1}^{t+1} 2^{-k}(-1)^k \tag{12}$$

$$= \frac{2^{t+1}-1}{3} + 2^{t+1} \cdot \left(-\frac{1}{2}\right) \cdot \frac{1-\left(-\frac{1}{2}\right)^{t+1}}{1-\left(-\frac{1}{2}\right)} \tag{13}$$

$$= \frac{2^{t+1}-1}{3} - \frac{1}{3} \cdot 2^{t+1}\left(1-(-2)^{-t-1}\right) \tag{14}$$

Since the base of N is always odd if N is not a power of 2, $(-2)^{-t-1} = 2^{-t-1}$ and this expression simplifies to

$$= \frac{2^{t+1}-1}{3} - \frac{2^{t+1}-1}{3} = 0 \tag{15}$$

□

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For example,

$$P_{820569}(-1) = 0$$

yet $c(820569) = 1230854 = 2 \cdot 615427$, which is not a power of two.

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For example,

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yet $c(820569) = 1230854 = 2 \cdot 615427$, which is not a power of two.

In fact, the following lemma implies that

$$P_N(-1) = 0$$

for every odd preimage N of $2^k \cdot 615427$ where k is odd.

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Lemma

If $P_{c^{-1}(N)}(-1) = 0$ *then* $P_{c^{-1}(4N)}(-1) = 0$*.*

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Lemma

If $P_{c^{-1}(N)}(-1) = 0$ then $P_{c^{-1}(4N)}(-1) = 0$.

Proof.

If $P_{c^{-1}(N)}(-1) = 0$ then $P_N(-1) = c^{-1}(N) = \frac{2N-1}{3}$. But then

$$
\begin{aligned}
P_{c^{-1}(4N)} &= c^{-1}(4N) - 4N + 2N - P_N(-1) && (16)\\
&= \frac{8N-1}{3} - \frac{12N}{3} + \frac{6N}{3} - \frac{2N-1}{3} && (17)\\
&= 0 && (18)
\end{aligned}
$$

□

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Finally, there exist N even such that $P_N(-1) = 0$.

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The least such example is

$$N = \quad 6094358 = 2 \cdot 83 \cdot 36713 \tag{19}$$

Another example is

$$N = 46507804 = 2^2 \cdot 7 \cdot 593 \cdot 2801 \tag{20}$$

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These two prime factorizations are apparently unrelated.

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1. Simple characterization of N such that $P_N(-1) = 0$

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2. Relationships between zeros of $P_N(z)$ and dynamics of $\{C^k(N)\}$

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2. Relationships between zeros of $P_N(z)$ and dynamics of $\{C^k(N)\}$
3. Applications of “factorization" of N based on linear factors of $P_N(z)$