Solutions to Dr. Z.'s Math 421(2), REAL Quiz #9

1. (5 points) Solve the boundary value pde problem:

$$u_{xx} = u_{tt} \quad ,0 < x < \pi \quad , \quad t > 0 \quad ;$$

$$u(0,t) = 0 \quad , \quad u(\pi,t) = 0 \quad , \quad t > 0 \quad ;$$

$$u(x,0) = \sin(7\pi x) \quad , \quad u_t(x,0) = \sin(8\pi x) \quad , \quad 0 < x < \pi \quad .$$

Sol. Note, this is a trick question! The "shortcut" method is not applicaple, since $\sin(7\pi x)$ and $\sin(8\pi x)$ are **not** of the form $\sin(nx)$ for **integer** n. Nevertheless, I gave three out of five points, to people who "used" the "shorcut method" (wrongly of course, since it is not applicable in this case), since I intended the initial conditions to be $u(x,0) = \sin(7x), u_t(x,0) = \sin(8x)$, and it was really a typo. For this problem the answer is simply (here $a = 1$)

$$u(x,t) = \sin(7x)\cos(7t) + \frac{\sin(8x)\sin(8t)}{8} \quad .$$

But the problem, as given still makes sense, except that it takes too long to do in five minutes. People who wrote

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(nx)\cos(nt) + \sum_{n=1}^{\infty} B_n \sin(nx)\sin(nt) \quad ,$$

where

$$A_n = \frac{2}{\pi}\int_0^{\pi} \sin(7\pi x)\sin nx\, dx \quad , \quad B_n = \frac{2}{n\pi}\int_0^{\pi} \sin(8\pi x)\sin nx\, dx$$

without actually computing these complicated and awkward integrals got full credit. (If you really want to do it, you use the product formula for $\sin A \sin B$ from the formula sheet, and then integrate the trig functions, getting an ugly mess for A_n and B_n.)

2. (5 points) Solve :

$$u_{xx} + u_{yy} = 0 \quad , \quad 0 < x < \pi \quad , \quad 0 < y < 1 \quad ,$$

Subject to

$$u(0,y) = 0 \quad , \quad u(\pi,y) = 0 \quad , \quad 0 < y < 1 \quad ;$$

$$u(x,0) = 0 \quad , \quad u(x,1) = 5 \quad , \quad 0 < x < \pi \quad .$$

Sol. There are eight families of product solutions

$$\cos(\lambda x)\cosh(\lambda y) \quad , \quad \cos(\lambda x)\sinh(\lambda y) \quad , \quad \sin(\lambda x)\cosh(\lambda y) \quad , \quad \sin(\lambda x)\sinh(\lambda y) \quad ,$$

and four others obtained by exchanging cos and cosh and sin and sinh. To make $u(0,y) = 0$ happy, we must rule out the first two, since plugging-in $x = 0$ does not yield 0. To male $u(x,0) = 0$ happy, we must rule out the third one, so all that remains is $u(x,y) = \sin(\lambda x)\sinh(\lambda y)$ (and its analog

$u(x,y) = \sinh(\lambda x)\sin(\lambda y))$. To make $u(\pi, y) = 0$ we need $u(\pi, y) = \sin(\lambda\pi)\sinh(\lambda y) = 0$. Since $\sinh(\lambda y)$ is not the zero function, we must insist that $\sin(\lambda\pi) = 0$ (for the analog $\sinh(\lambda\pi) = 0$, but this never happens, so we can forget about that option).

Solving the trig equation $\sin(\lambda\pi) = 0$ yields $\lambda\pi = n\pi$, solving for λ gives $\lambda = n$, n integer.

So the inifinte family $\{(\sin nx)(\sinh ny)\}$ are all solutions of the pde plus the first three conditions. By the superposition principle so is *any* combination

$$u(x,y) = \sum_{n=1}^{\infty} A_n (\sin nx)(\sinh ny) \quad ,$$

for **any** numbers $A_1, A_2, A_3, \ldots$.

It remains to make the last condition happy $u(x,1) = 5$. Plugging-in $y = 1$:

$$u(x,1) = \sum_{n=1}^{\infty} A_n (\sin nx)(\sinh n) = \sum_{n=1}^{\infty} (A_n \sinh n)(\sin nx)$$

So

$$5 = \sum_{n=1}^{\infty} (A_n \sinh n)(\sin nx) \quad .$$

So $A_n \sinh n$ are the **Fourier-Sine** coefficients of 5.

We have to find the **Fourier-Sine** series of 5 (no shortcuts are possible!) By the formula

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx$$

with

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx \quad ,$$

we get

$$a_n = \frac{2}{\pi}\int_0^{\pi} 5 \sin nx \, dx = \frac{10}{\pi}\int_0^{\pi} \sin nx \, dx = \left(\frac{10}{\pi}\right) \cdot \frac{(-\cos nx)}{n}\Big|_0^{\pi} = \frac{-10}{n\pi}(\cos 0 - \cos n\pi) = \frac{-10}{n\pi}(1-(-1)^n) \quad ,$$

so

$$A_n \sinh n = \frac{10}{n\pi}((-1)^n - 1) \quad ,$$

and solving for A_n we get

$$A_n = \frac{10}{n\pi sinh(n)}((-1)^n - 1).$$

Going back to $u(x,y)$ we get

$$u(x,y) = \sum_{n=1}^{\infty} \frac{10(1-(-1)^n)}{n sinh(n) \pi} (\sin nx)(\sinh ny) \quad .$$

This is the **answer**.

Comments: 1. This was a long problem, so people who wrote

$$A_n = \frac{10}{\pi sinh(n)} \int_0^{\pi} \sin nx \, dx \quad ,$$

also got full credit.