Solutions to Dr. Z.'s Math 421 REAL Quiz #8

1. Solve the boundary value problem

$$7\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad , \quad 0 < x < \pi \quad , \quad t > 0 \quad ,$$

subject to

$$u_x(0,t) = 0 \quad , \quad u_x(\pi,t) = 0 \quad , t > 0$$

$$u(x,0) = f(x) \quad , \quad 0 < x < \pi \quad ,$$

where

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < \pi/2; \\ 2, & \text{if } \pi/2 \le x < \pi; \end{cases}$$

(You may use the ready-made formula)

Sol.: Here $k = 7$,$L = \pi$ (so that we can use the simpler formula)

From the formula sheet: the solution is

$$u(x,t) = \frac{A_0}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} A_n e^{-kn^2 t} \cos nx \quad ,$$

where

$$A_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx \quad , \quad A_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx \quad .$$

Note: The boundary conditions are of the **insulating** kind, so we have a cosine-series, **not** a sine-series. Quite a few people used the formula for the boundary value problem $u(0,t) = 0, u(\pi,t) = 0$. Please read the question carefully, and only apply the relevant formula.

In this problem $k = 7$. First:

$$A_0 = \frac{2}{\pi}\int_0^{\pi} f(x) = \frac{2}{\pi}\int_{\pi/2}^{\pi} 2 = 2 \quad .$$

Next:

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx = \frac{2}{\pi}\int_0^{\pi/2} f(x)\cos nx + \frac{2}{\pi}\int_{\pi/2}^{\pi} f(x)\cos nx$$

$$= 0 + \frac{2}{\pi}\int_{\pi/2}^{\pi}(2)\cos nx = \frac{4}{\pi}\int_{\pi/2}^{\pi}\cos nx = (\frac{4}{\pi})\frac{\sin nx}{n}\Big|_{\pi/2}^{\pi}$$

$$= (\frac{4}{n\pi})(\sin n\pi - \sin n\pi/2) = -\frac{4\sin(n\pi/2)}{n\pi} \quad .$$

So we have

$$u(x,t) = 1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n} e^{-7n^2 t}\cos nx \quad .$$

This is an acceptable **answer**. It can be made simpler as follows. When n is even $\sin(n\pi/2)$ equals 0, and when n is odd, $n = 2k+1$, it equals $(-1)^k$, so we can get a better solution:

$$u(x,t) = 1 - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} e^{-7(2k+1)^2 t} \cos(2k+1)x \quad .$$

Comment: Quite a few people "simplified" $\sin(n\pi/2)$ in a wrong way (for example $(-1)^n$). Be careful. When in doubt, do not simplify. When you have a proposed simplification, just check it for a few values of n to make sure that it is correct.