Solutions to Dr. Z.'s Math 421(1) REAL Quiz #6

1. (a) (8 points) Find the complex Fourier series $f(x) = x$ on the interval $-\pi < x < \pi$. (b) (2 points) Find its frequency spectrum.

Sol. to a):

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx}\, dx \quad .$$

When $n = 0$, this is easy:

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x\, dx = \frac{1}{2\pi} \left(\frac{x^2}{2} \Big|_{-\pi}^{\pi} \right) = \frac{1}{2\pi} \left(\frac{\pi^2 - (-\pi)^2}{2} \right) = 0 \quad .$$

Remember (integration by parts, or use the cheatsheet)

$$\int x e^{cx}\, dx = \left(\frac{-1}{c^2} + \frac{x}{c} \right) e^{cx}$$

Here $c = -in$ $(n \neq 0)$ so:

$$\int_{-\pi}^{\pi} x e^{-inx}\, dx = \left(\frac{-1}{(-in)^2} + \frac{x}{-in} \right) e^{-inx} \Big|_{-\pi}^{\pi} = \left(\frac{-1}{(-in)^2} + \frac{ix}{n} \right) e^{-inx} \Big|_{-\pi}^{\pi}$$

$$\left(\frac{1}{n^2} + \frac{i\pi}{n} \right) e^{-in(\pi)} - \left(\frac{1}{n^2} + \frac{i(-\pi)}{n} \right) e^{in(\pi)}$$

Now $e^{i\pi} = -1$, so $e^{in\pi} = (-1)^n$. and also $e^{-in\pi} = (-1)^n$. So this equals

$$\left(\frac{1}{n^2} + \frac{i\pi}{n} \right) (-1)^n - \left(\frac{1}{n^2} + \frac{i(-\pi)}{n} \right) (-1)^n = \frac{2i\pi(-1)^n}{n} \quad .$$

Multiplying by $\frac{1}{2\pi}$, we get:

$$c_n = \frac{1}{2\pi} \frac{2i\pi(-1)^n}{n} = \frac{i(-1)^n}{n}$$

Ans. to 1a: The Fourier Series of $f(x) = x$ on the interval $(-\pi, \pi)$ is:

$$\sum_{n=-\infty, n\neq 0}^{\infty} \frac{i(-1)^n}{n} e^{inx} \quad .$$

Comment: Most people stated it right, but only about %40 got it fully correct. Some people need to review their algebra, and how to handle complex numbers. Remember: $i^2 = -1, i^3 = -i, i^4 = 1, 1/i = -i$.

Sol. of 1b: Recall that the *frequency spectrum* of a function $f(x)$ on the interval $(-p, p)$ is the infinite (double) sequence of points:

$$(n\omega, |c_n|) \quad , n = 0, \pm 1, \pm 2, \ldots \quad .$$

Where $\omega = \pi/p$.

Here $p = \pi$ so $\omega = \pi/p = \pi/\pi = 1$, so the **frequency spectrum** is

$$(n, |c_n|), n = 0, \pm 1, \pm 2, \pm 3, \quad .$$

When $n \neq 0$, $c_n = \frac{i(-1)^n}{n}$, so $|c_n| = \frac{1}{|n|}$, (since $|i| = 1$ and $|(-1)^n| = 1$). When $n = 0$, $c_0 = 0$, so we get $(0, 0)$.

So **Ans. to 1b**: The frequency spectrum of $f(x) = x$ on the interval $(-\pi, \pi)$ is the set

$$\{(0,0)\} \cup \{(n, \frac{1}{|n|})\,;\, n = \pm 1, \pm 2, \pm 3, \quad \} \quad .$$

Comment: A common error was to get ω wrong. Some people got $\omega = 2$. Watch out! Another common error was to write $(n, 1/n)$. This is wrong for two reasons. First the answer is $(n\omega, |c_n|)$. Don't forget the absolute value. Since n can be positive or negative and the second component of the energy spectrum is never negative it should be $(n, |1/n|)$. But even this is nonsense when $n = 0$, that has to be done separately. Since $c_n = 0$, one has to add the point $(0, 0)$.