Solutions to Dr. Z.'s Math 421(1) Quiz #5

1. Find the Fourier series of

$$f(x) = \begin{cases} 2, & \text{if } -\pi < x < 0; \\ -1, & \text{if } 0 \le x < \pi. \end{cases}$$

Sol. The Fourier Series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ,$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx \quad , \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \quad , \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \quad .$$

Let's do the three integrations.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_{-\pi}^{0} f(x)\, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x)\, dx =$$

$$\frac{1}{\pi} \int_{-\pi}^{0} 2 + \frac{1}{\pi} \int_{0}^{\pi} (-1) = \frac{1}{\pi} 2\pi + \frac{1}{\pi}(-\pi) = 2 - 1 = 1 \quad .$$

So $a_0 = 1$.

Next:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_{-\pi}^{0} f(x) \cos nx\, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x) \cos nx\, dx =$$

$$\frac{1}{\pi} \int_{-\pi}^{0} 2 \cos nx + \frac{1}{\pi} \int_{0}^{\pi} (-1) \cos nx = \frac{2}{\pi} \frac{\sin nx}{n} \Big|_{-\pi}^{0} - \frac{1}{\pi} \frac{\sin nx}{n} \Big|_{0}^{\pi}$$

$$\frac{2}{\pi} \frac{\sin 0 - \sin(-\pi n)}{n} - \frac{1}{\pi} \frac{\sin \pi n - \sin 0}{n} = 0 \quad .$$

So $a_n = 0$. Next:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{1}{\pi} \int_{-\pi}^{0} f(x) \sin nx\, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x) \sin nx\, dx =$$

$$\frac{1}{\pi} \int_{-\pi}^{0} 2 \sin nx + \frac{1}{\pi} \int_{0}^{\pi} (-1) \sin nx = \frac{2}{\pi} \cdot \frac{-\cos nx}{n} \Big|_{-\pi}^{0} - \frac{1}{\pi} \cdot \frac{-\cos nx}{n} \Big|_{0}^{\pi}$$

$$\frac{-2}{\pi} \cdot \frac{\cos 0 - \cos(-n\pi)}{n} + \frac{1}{\pi} \cdot \frac{\cos n\pi - \cos 0}{n} = \frac{-2}{\pi} \cdot \frac{1 - (-1)^n}{n} + \frac{1}{\pi} \cdot \frac{(-1)^n - 1}{n} = \frac{-2 + 2(-1)^n}{n\pi} + \frac{(-1)^n - 1}{n\pi}$$

$$= \frac{-3 + 3(-1)^n}{n\pi} = \frac{-3(1 - (-1)^n)}{n\pi} \quad .$$

Combining we get **First Ans.** The Fourier series of $f(x)$ is

$$f(x) = \frac{1}{2} - 3 \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n\pi} \sin nx \quad .$$

But when n is even, $1-(-1)^n = 0$ so let's write $n = 2k+1$ $(k = 0, 1, \ldots)$, and note that when n is odd $1-(-1)^n = 2$ so we have **Better Ans.** The Fourier series of $f(x)$ is

$$f(x) = \frac{1}{2} - \frac{6}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)} \sin(2k+1)x \quad .$$

Comment: People who gave the first (unsimplified) answer still got full credit. Unless it states "simplify as much as possible", the former answer is acceptable.