Solutions to the Attendance Quiz for the Linear Algebra Class

1. Find all the eigenvalues of the matrix

$$\begin{bmatrix} -3 & -4 \\ 12 & 11 \end{bmatrix} \quad ,$$

and determine a basis for each eigenspace.

Sol.: The **Characteristic matrix** is

$$\begin{bmatrix} -3-\lambda & -4 \\ 12 & 11-\lambda \end{bmatrix} \quad .$$

Taking its **determinant**, we have

$$\det \begin{bmatrix} -3-\lambda & -4 \\ 12 & 11-\lambda \end{bmatrix} = (-3-\lambda)(11-\lambda)-(-4)(12) = (\lambda+3)(\lambda-11)+48 = \lambda^2-8\lambda-33+48 = \lambda^2-8\lambda+15$$

So the **characteristic equation** is:

$$\lambda^2 - 8\lambda + 15 = 0 \quad .$$

Factorizing we get

$$(\lambda - 3)(\lambda - 5) = 0 \quad .$$

So the **eigenvalues** are $\lambda = 3$ and $\lambda = 5$.

For each of them we must still find a basis for the **eigenspace**.

For $\lambda = 3$, we have to find vector(s) $\begin{bmatrix} a \\ b \end{bmatrix}$, such that

$$\begin{bmatrix} -3 & -4 \\ 12 & 11 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 3 \begin{bmatrix} a \\ b \end{bmatrix} \quad .$$

In everyday notation:

$$-3a - 4b = 3a \quad , \quad 12a + 11b = 3b \quad .$$

Rearranging:

$$-6a - 4b = 0 \quad , \quad 12a + 8b = 0 \quad .$$

The second equation is twice the first one, so we can ignore it, and we get $b = -\frac{6}{4}a = -\frac{3}{2}a$. To make it nice, we can take a to be any non-zero number. Taking $a = 2$ gives $b = -3$, so a basis vector is $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

For $\lambda = 5$, we have to find vector(s) $\begin{bmatrix} a \\ b \end{bmatrix}$, such that

$$\begin{bmatrix} -3 & -4 \\ 12 & 11 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 5 \begin{bmatrix} a \\ b \end{bmatrix} \quad .$$

In everyday notation:

$$-3a - 4b = 5a \quad , \quad 12a + 11b = 5b \quad .$$

Rearranging:

$$-8a - 4b = 0 \quad , \quad 12a + 6b = 0 \quad .$$

The second equation is a multiple of the first one, so we can ignore it, and we get $b = -\frac{8}{4}a = -2a$. To make it nice, we can take a to be any non-zero number. Taking $a = 1$ gives $b = -2$, so a basis vector is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Ans. The eigenvalues are $\lambda = 3$ and a basis for its eigenspace is $\{\begin{bmatrix} 2 \\ -3 \end{bmatrix}\}$, and $\lambda = 5$ and a basis for its eigenspace is $\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\}$.