Solutions to the Attendance Quiz for Lecture 8

1 Find the Fourier series of $f(x) = 2x$ on the interval $(-2\pi, 2\pi)$.

First Sol. (By transfroming to the interval $(-\pi, \pi)$ that has nicer-looking formulass)

We transform the problem to the standard interval, $(-\pi, \pi)$, by considering

$$g(x) = f(2x) \quad ,$$

that is definied on $(-\pi, \pi)$. At the end, once we get the answer for $g(x)$, we go back to $f(x)$ with the reverse relation

$$f(x) = g(x/2) \quad .$$

In this problem

$$g(x) = f(2x) = 2(2x) = 4x \quad .$$

We use the formulas

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad ,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad ,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad .$$

Since $f(x)$ is an **odd** function, a_n is **automatically** 0, so we shouldn't bother with it. Also a_0 is 0. We only have to worry about b_n.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} 4x\sin nx\,dx \quad .$$

Now we use the formula from the formula sheet

$$\int x\sin ax = \frac{\sin ax - ax\cos ax}{a^2} \quad ,$$

So

$$b_n = \frac{4}{\pi}\cdot\int_{-\pi}^{\pi} x\sin nx\,dx = \frac{4}{\pi}\cdot\frac{\sin nx - nx\cos nx}{n^2}\Big|_{-\pi}^{\pi}$$

$$= \frac{4}{\pi}\cdot\frac{\sin(n\pi) - n\pi\cos(n\pi)}{n^2} - \frac{4}{\pi}\cdot\frac{\sin(n(-\pi)) - n(-\pi)\cos(n(-\pi))}{n^2}$$

$$= -\frac{4n(-1)^n}{n^2} - \frac{4n(-1)^n}{n^2} = -\frac{4(-1)^n}{n} - \frac{4(-1)^n}{n} = -\frac{8(-1)^n}{n} \quad .$$

So the Fourier Series of $g(x)$ is

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx = 0 + 0 + \sum_{n=1}^{\infty} -\frac{8(-1)^n}{n}\sin nx = -8\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx \quad .$$

Finally: going back to $f(x)$, using $f(x) = g(x/2)$, we get that the Fourier Series of $f(x) = 2x$ is:

$$-8\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{nx}{2} \quad .$$

Ans. to 1.: The Fourier Series of $f(x) = 2x$ over the interval $(-2\pi, 2\pi)$ is $-8\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{nx}{2}$.

Second Sol. (By using the more complicated formulas for a general interval $(-L, L)$).

We use the formulas

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx \quad ,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi}{L}x)\,dx \quad ,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x)\,dx \quad ,$$

Since $f(x)$ is an **odd** function, a_n is **automatically** 0, so we shouldn't bother with it. Also (once again because $f(x)$ is odd) a_0 is 0. So we only have to worry about b_n.

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\sin(\frac{n\pi}{L}x)dx = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} 2x\,\sin(\frac{n\pi}{2\pi}x)dx = \frac{1}{\pi}\int_{-2\pi}^{2\pi} x\,\sin(\frac{n}{2}x)dx$$

Now we use the formula from the formula sheet

$$\int x\sin ax = \frac{\sin ax - ax\cos ax}{a^2} \quad ,$$

So

$$b_n = \frac{1}{\pi}\left(\frac{\sin((n/2)x) - (n/2)x\cos((n/2)x)}{(n/2)^2}\right)\Big|_{-2\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left(\frac{\sin((n/2)(2\pi)) - (n/2)(2\pi)\cos((n/2)(2\pi))}{(n/2)^2}\right) - \frac{1}{\pi}\left(\frac{\sin(n/2)(-2\pi) - (n/2)(2\pi)\cos((n/2)(-2\pi))}{(n/2)^2}\right)$$

$$\frac{1}{\pi}\left(\frac{\sin(n\pi) - (n/2)(2\pi)\cos((n\pi))}{(n/2)^2}\right) - \frac{1}{\pi}\left(\frac{\sin(-n\pi) - (n/2)(2\pi)\cos((-n\pi))}{(n/2)^2}\right)$$

$$\frac{1}{\pi}\left(\frac{0 - n\pi(-1)^n)}{(n/2)^2}\right) - \frac{1}{\pi}\left(\frac{0 - n\pi(-1)^n}{(n/2)^2}\right)$$

$$-\frac{4}{\pi n^2}(n\pi)(-1)^n) - \frac{4}{\pi n^2}(n\pi)(-1)^n) = \frac{-8}{n}\cdot(-1)^n)$$

So, once again

Ans. to 1.: The Fourier Series of $f(x) = 2x$ over the interval $(-2\pi, 2\pi)$ is

$$-8\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(\frac{n}{2}x) \quad .$$

Comment: Few people got it completely, but many were on the right track. This is an important type of problem. Please study it carefully.