Solutions to the Attendance Quiz for Lecture 6

1.

$$\frac{dx}{dt} = -x + y \quad ,$$

$$\frac{dy}{dt} = 2x \quad ,$$

$$x(0) = 0 \quad , \quad y(0) = 3 \quad .$$

Sol.: We first apply the Laplace transform, and a always, write $X = \mathcal{L}\{x\}, Y = \mathcal{L}\{y\}$.

$$\mathcal{L}\{x'(t)\} = \mathcal{L}\{-x + y\} \quad ,$$

$$\mathcal{L}\{y'(t)\} = \mathcal{L}\{2x\} \quad .$$

Since $\mathcal{L}\{x'\} = sX - x(0) = sX$ and $\mathcal{L}\{y'\} = sY - y(0) = sY - 3$, we have:

$$sX = -X + Y \quad ,$$

$$sY - 3 = 2X \quad .$$

Rearranging in **standard form**:

$$(s+1)X - Y = 0 \quad ,$$

$$-2X + sY = 3 \quad .$$

From the first equation, we have $Y = (s+1)X$. Plugging this into the second equation we get

$$-2X + s(s+1)X = 3 \quad .$$

Factoring out X:

$$(s(s+1) - 2)X = 3 \quad ,$$

Expanding:

$$(s^2 + s - 2)X = 3 \quad ,$$

Factorizing:

$$(s-1)(s+2)X = 3 \quad ,$$

Solving for X:

$$X = \frac{3}{(s-1)(s+2)} \quad .$$

Back-substitution:

$$Y = (s+1)X = \frac{3(s+1)}{(s-1)(s+2)} \quad .$$

Now it is time to apply $\mathcal{L}^{-1}$.

$$x(t) = \mathcal{L}^{-1}\{\frac{3}{(s-1)(s+2)}\} \quad , \quad y(t) = \mathcal{L}^{-1}\{\frac{3s+3}{(s-1)(s+2)}\} \quad .$$

Partial fraction for X, we use the following **template**

$$\frac{3}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2} \quad .$$

Common denominator:

$$\frac{3}{(s-1)(s+2)} = \frac{A(s+2)+B(s-1)}{(s-1)(s+2)} \quad .$$

Equating the tops: $3 = A(s+2)+B(s-1)$. Convenient values: $s=-2$ gives $3 = A(-2+2)+B(-2-1)$ so $B=-1$; $s=1$ gives $3 = A(1+2)+B(1-1)$, so $3=3A$, so $A=1$, going back to the template:

$$\frac{3}{(s-1)(s+2)} = \frac{1}{s-1} - \frac{1}{s+2} \quad .$$

And, so

$$x(t) = \mathcal{L}^{-1}\{\frac{1}{s-1} - \frac{1}{s+2}\} = e^t - e^{-2t} \quad .$$

Now we do the same thing for $Y = \frac{3(s+1)}{(s-1)(s+2)}$. We must use the following **template**

$$\frac{3s+3}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2} \quad .$$

Common denominator:

$$\frac{3s+3}{(s-1)(s+2)} = \frac{A(s+2)+B(s-1)}{(s-1)(s+2)} \quad .$$

Equating the tops: $3s+3 = A(s+2)+B(s-1)$. Convenient values: $s=-2$ gives $3(-2)+3 = A(-2+2)+B(-2-1)$ so $B=1$; $s=1$ gives $3(1)+3 = A(1+2)+B(1-1)$, so $6=3A$, so $A=2$, going back to the template:

$$\frac{3}{(s-1)(s+2)} = \frac{2}{s-1} + \frac{1}{s+2} \quad .$$

And, so

$$y(t) = \mathcal{L}^{-1}\{\frac{2}{s-1} + \frac{1}{s+2}\} = 2e^t + e^{-2t} \quad .$$

Ans. to 1: $x(t) = e^t - e^{-2t}$, $y(t) = 2e^t + e^{-2t}$. .

Comment: About %40 of the students did it perfectly. Another %40 know how to do it, but messed up sooner or later, or ran out of time.