Solution to the Attendance Quiz for Lecture 5

1. Solve the initial-value problem

$$y'' + y = \delta(t - 2\pi) \quad , \quad y(0) = 0 \quad y'(0) = 1 \quad .$$

Sol.: First, we apply $\mathcal{L}$:

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\delta(t - 2\pi)\} \quad .$$

Using $\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$ we have $\mathcal{L}\{y''\} = s^2Y - s \cdot 0 - 1 = s^2Y - 1$. Of course, $\mathcal{L}\{y\} = Y$, and $\mathcal{L}\{\delta(t - 2\pi)\} = e^{-2\pi s}$. We get

$$s^2Y - 1 + Y = e^{-2\pi s} \quad ,$$

Solving for Y:

$$(s^2 + 1)Y = 1 + e^{-2\pi s} \quad ,$$

$$Y = \frac{1}{s^2 + 1} + \frac{e^{-2\pi s}}{s^2 + 1} \quad .$$

Applying $\mathcal{L}^{-1}$, we get

$$y(t) = \mathcal{L}^{-1}\{\frac{1}{s^2 + 1}\} + \mathcal{L}^{-1}\{\frac{e^{-2\pi s}}{s^2 + 1}\} \quad .$$

The first piece is a simple table look-up:

$$\mathcal{L}^{-1}\{\frac{1}{s^2 + 1}\} = \sin t \quad .$$

For the second, we need the general formula

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t - a)\mathcal{U}(t - a) \quad .$$

Here $a = 2\pi$ and $F(s) = \frac{1}{s^2+1}$. We already know that $f(t) = \sin t$, so

$$\mathcal{L}^{-1}\{e^{-2\pi s}\frac{1}{s^2 + 1}\} = \sin(t - 2\pi)\mathcal{U}(t - 2\pi) \quad .$$

Combining, we get

$$y(t) = \sin t + \sin(t - 2\pi)\mathcal{U}(t - 2\pi) \quad .$$

This is a **correct** answer, but using elementary trig, this can be simplified, since $\sin(t - 2\pi) = \sin t$. So we have:

Ans. to 1: $y(t) = \sin t + \sin t\, \mathcal{U}(t - 2\pi)$.