Solutions to Attendance Quiz for Lecture 3

1. Use the Laplace Transform to solve the following initial-value problem.

$$y' + y = f(t), y(0) = 0, where f(t) = \begin{cases} 3, & \text{if } 0 \le t < 2; \\ -1, & \text{if } 2 \le t < \infty. \end{cases} \quad ,$$

Solution to 1.: We first write $f(t)$ in terms of $\mathcal{U}(t)$.

$f(t) = 3(\mathcal{U}(t-0)-\mathcal{U}(t-2))+(-1)(\mathcal{U}(t-2)-\mathcal{U}(t-\infty) = 3(1-\mathcal{U}(t-2))-(\mathcal{U}(t-2)-0) = 3-4\mathcal{U}(t-2)$

So we have to solve the IVP

$$y' + y = 3 - 4\mathcal{U}(t-2) \quad y(0) = 0 \quad .$$

Apply $\mathcal{L}$:

$$\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(3) - 4\mathcal{L}(\mathcal{U}(t-2))$$

As usual, set $Y = \mathcal{L}(y)$, so $\mathcal{L}(y') = sY - y(0) = sY$, and we have

$$sY + Y = 3\mathcal{L}(1) - 4\mathcal{L}(\mathcal{U}(t-2)) = \frac{3}{s} - \frac{4e^{-2s}}{s}$$

$$Y = \frac{3}{s(s+1)} - \frac{4e^{-2s}}{s(s+1)} \quad .$$

This is the half-way point. Now it is time to apply $\mathcal{L}^{-1}$. Calling $F(s) = \frac{1}{s(s+1)}$, we have, using partial fractions (you do it!)

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1} \quad ,$$

so

$$f(t) = 1 - e^{-t} \quad .$$

So

$$\mathcal{L}^{-1}(\frac{3}{s(s+1)}) = 3(1 - e^{-t})$$

$$\mathcal{L}^{-1}(\frac{4e^{-2s}}{s(s+1)}) = 4(1 - e^{-(t-2)})\mathcal{U}(t-2) \quad .$$

Combining, we get **Ans.**:

$$y(t) = 3(1 - e^{-t}) - 4(1 - e^{-(t-2)})\mathcal{U}(t-2) \quad .$$

Comments: Many people got toe half-way point, but very few finished it, since I didn't give you enough time, and didn't have time to properly cover it in class. I will try to go over it briefly next time.