Solutions to Attendance Quiz for Lecture 21

1.: Solve the pde

$$u_{xx} = u_t \quad , \quad , -\infty < x < \infty \quad , \quad t > 0$$

subject to the initial condition

$$u(x,0) = \begin{cases} 0, & \text{if } -\infty < x < -2; \\ 1, & \text{if } -2 \le x < 0; \\ 2, & \text{if } 0 \le x < 3; \\ 0, & \text{if } 3 \le x < \infty; \end{cases} \quad .$$

Sol.: Applying the **Fourier transform** (w.r.t to x) to the pde $u_{xx} = u_t$, We get

$$\mathcal{F}(u_{xx}) = \mathcal{F}(u_t)$$

,

Let, as usual, $U(\alpha,t) := \mathcal{F}(u(x,t))$. Using $\mathcal{F}\{f^{(n)}(x)\} = (-i\alpha)^n F(\alpha)$, we get

$$(-i\alpha)^2 U(\alpha,t) = \frac{d}{dt} U(\alpha,t) \quad .$$

This is a very simple ode in t

$$\frac{d}{dt} U(\alpha,t) = -\alpha^2 U(\alpha,t) \quad ,$$

whose general solution is

$$U(\alpha,t) = c\, e^{-\alpha^2 t} \quad ,$$

where c is a constant (w.r.t. t) to be determined. plugging-in $t = 0$ we get

$$c = U(\alpha,0) \quad .$$

But $U(\alpha,0)$ is the Fourier transform of $u(x,0)$, so

$$U(\alpha,0) = \int_{-2}^{0} 1 \cdot e^{i\alpha x}\, dx + \int_{0}^{3} 2 \cdot e^{i\alpha x}\, dx$$

$$= \frac{e^{i\alpha x}}{i\alpha}\Big|_{-2}^{0} + 2\,\frac{e^{i\alpha x}}{i\alpha}\Big|_{0}^{3}$$

$$= \frac{1 - e^{-2i\alpha}}{i\alpha} + 2\frac{e^{3i\alpha} - 1}{i\alpha} = \frac{-1 + 2e^{3i\alpha} - e^{-2i\alpha}}{i\alpha}$$

Hence we have an explicit expression for $U(\alpha,t)$:

$$U(\alpha,t) \;=\; \frac{-1 + 2e^{3i\alpha} - e^{-2i\alpha}}{i\alpha} e^{-\alpha^2 t} \quad ,$$

Finally we take the **intverse Fourier transrom** and get

Answer:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-1 + 2e^{3i\alpha} - e^{-2i\alpha}}{i\alpha}\, e^{-\alpha^2 t - i\alpha x}\, d\alpha \quad ,$$