Solutions to the Attendance Quiz 1

1. Using the **definition** find the Laplace transform $\mathcal{L}\{f(t)\}$ (alias $F(s)$) of

$$f(t) = \begin{cases} 3, & \text{if } 0 \le t \le 1; \\ e^t, & \text{if } t \ge 1. \end{cases}$$

Sol.: By **definition**

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \quad .$$

Since this is a **discontinuous** function defined differently in different intervals, we have to **break-up** the integration from 0 to ∞ to the two parts: from 0 to 1, and from 1 to ∞:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st} f(t)\, dt + \int_1^\infty e^{-st} f(t)$$

Now, at each of the two integrals, we replace $f(t)$ by the appropriate expression:

$$= \int_0^1 e^{-st} \cdot 3\, dt + \int_1^\infty e^{-st} \cdot e^t \quad .$$

The first integral is a **proper** integral, since the limits of integration are **finite**. Using the famous formula

$$\int e^{ct}\, dt = \frac{e^{ct}}{c} + C \quad ,$$

we have

$$\int_0^1 e^{-st} \cdot 3\, dt = 3 \int_0^1 e^{-st}\, dt = \frac{3e^{-st}}{-s}\Big|_0^1 = \frac{-3}{s}(e^{-s\cdot 1} - e^{-s\cdot 0}) = \frac{-3}{s}(e^{-s} - 1) = \frac{3(1-e^{-s})}{s} \quad .$$

Now we go to the second, **improper**, integral: (assume that $s > 1$, as we may)

$$\int_1^\infty e^{-st} \cdot e^t = \int_1^\infty e^{(1-s)t} = \frac{e^{(1-s)t}}{1-s}\Big|_1^\infty$$

$$= \frac{e^{(1-s)\cdot\infty}}{1-s} - \frac{e^{(1-s)\cdot 1}}{1-s} = \frac{e^{-\infty}}{1-s} - \frac{e^{(1-s)}}{1-s} = 0 - \frac{e^{(1-s)}}{1-s} = \frac{e^{(1-s)}}{s-1} \quad .$$

Adding the two pieces up, we get

Ans. to 1: $\mathcal{L}\{f(t)\} = \frac{3(1-e^{-s})}{s} + \frac{e^{(1-s)}}{s-1} \quad .$

Comments: Only about %30 of the students finished it completely correctly, but about %80 were on the right track. Many people had trouble with the improper integral.

2. Using Tables, find $\mathcal{L}\{f(t)\}$, if $f(t) = (t+1)(t+2) + e^t + \sin t$.

Sol.: First we **expand**:

$$f(t) = t^2 + 3t + 2 + e^t + \sin t \quad .$$

Next we apply $\mathcal{L}$:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 + 3t + 2 + e^t + \sin t\} \quad .$$

Using the **linearity** property of $\mathcal{L}$, this becomes

$$\mathcal{L}\{t^2\} + 3\mathcal{L}\{t\} + 2\mathcal{L}\{1\} + \mathcal{L}\{e^t\} + \mathcal{L}\{\sin t\} \quad .$$

Now (and only now!) we use the table:

$$\mathcal{L}\{t^k\} = \frac{k!}{t^{k+1}}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad ,$$

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2 + k^2} \quad .$$

So

$$\mathcal{L}\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3} \quad ,$$

$$\mathcal{L}\{t\} = \frac{1!}{s^{1+1}} = \frac{1}{s^2} \quad ,$$

$$\mathcal{L}\{1\} = \frac{0!}{s^{0+1}} = \frac{1}{s} \quad ,$$

$$\mathcal{L}\{e^t\} = \frac{1}{s-1} \quad ,$$

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1^2} = \frac{1}{s^2 + 1} \quad .$$

Putting these together, we get: **Ans. to 2.**: $\mathcal{L}\{f(t)\} = \frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s} + \frac{1}{s-1} + \frac{1}{s^2+1} \quad .$

Comment: About %80 of the students got it completely right. Some people forgot what $k!$ means! $k!$ means $1 \cdot 2 \cdots k$, so 1! is 1, 2! is 2, 3! is 6 etc. A couple of people had k in the answer. This is wrong! k only features in the table, and one has to find the applicable k