Solutions to the Attendance Quiz for Lecture 19

Version of Dec. 19, 2024 (thanks to Daryl Chu)

1.: Find the Laplace Transform of the pde $u_{xx} = 4u_{tt}$, $t > 0$.

Sol. of 1: Let $U(x,s) = \mathcal{L}\{u(x,t)\}$, and apply $\mathcal{L}$ to the pde getting

$$\mathcal{L}\{u_{xx}\} = 4\mathcal{L}\{u_{tt}\}$$

So

$$U''(x,s) = 4(s^2U(x,s) - su(x,0) - u_t(x,0)) = 4s^2U(x,s) - 4su(x,0) - 4u_t(x,0))$$

Cleaning up, we get the ode

$$U''(x,s) \; - \; 4s^2U(x,s) \; = \; -4su(x,0) - 4u_t(x,0) \quad .$$

2.: Solve the pde

$$u_{xx} = u_{tt} \quad , \quad 0 < x < 2 \quad , \quad t > 0 \quad ,$$

subject to the **boundary-conditions**

$$u(0,t) = 0 \quad , \quad u(2,t) = 0 \quad , \quad t > 0 \quad ,$$

and the **initial conditions**

$$u(x,0) = 0 \quad , \quad u_t(x,0) = \sin(\pi x/2) \quad , \quad 0 < x < 2 \quad .$$

Sol. to 2: Let $U(x,s) = \mathcal{L}\{u(x,t)\}$. Apply $\mathcal{L}$ to the pde

$$\mathcal{L}\{u_{xx}\} = \mathcal{L}\{u_{tt}\} \quad .$$

Using the 'dictionary':

$$U''(x,s) = s^2U(x,s) - su(x,0) - u_t(x,0) \quad .$$

Using the initial conditions, we get the **inhomogeneous** ode (for the sake of brevity we rewrite $U(x,s)$ as $U(x)$

$$U''(x) \; - \; s^2U(x) = -\sin(\pi x/2) \quad .$$

Translating the boundary conditions gives

$$U(0) = 0 \quad , \quad U(2) = 0 \quad .$$

The general solution of the *inhomogeneous version* is:

$$U''(x) \; - \; s^2U(x) = 0 \quad .$$

is

$$U(x) = c_1 e^{sx} + c_2 e^{-sx}$$

(Since the *auxiliary equation* is $r^2 - s^2 = 0$ whose roots are $r = -s$ and $r = s$).

A **template** for a **particular solution**, in generatl would be

$$U(x) = A\sin(\pi x/2) + B\cos(\pi x/2) \quad ,$$

but since $U'(x)$ does not show up, it is safe to take the simpler template

$$U(x) = A\sin(\pi x/2) \quad .$$

So $U''(x) = -A(\frac{\pi}{2})^2 \sin(\pi x/2)$. Plugging into the ode, we get

$$-A(\frac{\pi}{2})^2 \sin(\pi x/2) - s^2 A \sin(\pi x/2) = -\sin(\pi x/2)$$

Simplifying:

$$-A\sin(\pi x/2)((\frac{\pi}{2})^2 + s^2) = -\sin(\pi x/2)$$

Dividing by $-\sin(\pi x/2)$:

$$A(\frac{\pi}{2})^2 + s^2) = 1 \quad ,$$

yielding:

$$A = \frac{1}{(\pi/2)^2 + s^2} \quad .$$

So the **general solution** of the ode is

$$U(x) = c_1 e^{sx} + c_2 e^{-sx} + \frac{1}{(\pi/2)^2 + s^2} \sin(\frac{\pi}{2} x) \quad .$$

In order to find c_1 and c_2, we must use the boundary condition $U(0) = 0$ and $U(2) = 0$, getting:

$$c_1 + c_2 = 0 \quad , \quad c_1 e^{2s} + c_2 e^{-2s} = 0 \quad ,$$

whose solution is $c_1 = 0$ and $c_2 = 0$. Going back to the general solution, we found a formula for $U(x,s)$:

$$U(x,s) = \frac{1}{(\pi/2)^2 + s^2} \sin(\frac{\pi}{2} x) \quad .$$

Finally, we apply $\mathcal{L}^{-1}$ getting (using the fact from the table that $\mathcal{L}^{-1}\{\frac{1}{s^2+k^2}\} = \frac{1}{k}\sin kt$):

$$u(x,t) = \mathcal{L}^{-1}\{\frac{1}{(\pi/2)^2 + s^2}\} = \sin(\frac{\pi}{2}x)\mathcal{L}^{-1}\{\frac{1}{(\pi/2)^2 + s^2}\} = \sin(\frac{\pi}{2}x)\frac{\sin(\frac{\pi}{2}t)}{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\sin(\frac{\pi}{2}x)\sin(\frac{\pi}{2}t) \quad .$$

Ans. to 2: $u(x,t) = \frac{2}{\pi}\sin(\frac{\pi}{2}x)\sin(\frac{\pi}{2}t)$.