Solution to the Attendance Quiz for Lecture 17

1. Solve :

$$u_{xx} + u_{yy} = 0 \quad , \quad 0 < x < \pi \quad , \quad 0 < y < 1 \quad ,$$

Subject to

$$u(0, y) = 0 \quad , \quad u(\pi, y) = 0 \quad , \quad 0 < y < 1 \quad ;$$

$$u(x, 0) = 0 \quad , \quad u(x, 1) = f(x) \quad , \quad 0 < x < \pi \quad .$$

Sol.: Unlike the Heat and Wave Equations, where you are allowed to use canned formulas (unless specified otherwise), for Laplace's Equation, you are supposed to do it **from scratch**, using **separation of variables** and Fourier Series.

We first introduce the **template**

$$u(x, y) = X(x)Y(y) \quad .$$

So

$$u_{xx} = X''Y \quad , u_{yy} = XY'' \quad .$$

Putting this into the pde:

$$X''Y + XY'' = 0 \quad .$$

Dividing by XY:

$$\frac{X''Y + XY''}{XY} = 0 \quad .$$

Simplifying:

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \quad .$$

Moving the Y stuff to the right side:

$$\frac{X''}{X} = -\frac{Y''}{Y} \quad .$$

Or, in long-hand

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} \quad .$$

As usual, in this method, we say that the left-side does **not** depend on y, while the right-side does **not** depend on x. But they are equal to each other, so **neither** of them depends on either x, or y. In other words, both sides are equal to the **same** constant. That constant can be either positive, zero, or negative. It turns out that the positive case would only give you the trivial (zero) solution, so let's call that constant $-\lambda^2$.

We have to solve **two odes** :

$$\frac{X''(x)}{X(x)} = -\lambda^2 \quad .$$

$$-\frac{Y''(y)}{Y(y)} = -\lambda^2 \quad .$$

Cleaning up:

$$X''(x) + \lambda^2 X(x) = 0 \quad .$$

$$Y''(y) - \lambda^2 Y(y) = 0 \quad .$$

The **general solution** of the $X(x)$ equation is

$$c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$$

The **general solution** of the $Y(y)$ equation is

$$c_3 \cosh(\lambda y) + c_4 \sinh(\lambda y)$$

(Note that in this business we use this form rather than the more customary form $c_3 e^{\lambda x} + c_4 e^{-\lambda x}$.) So the product solution, going back to $u(x, y) = X(x)Y(y)$ is:

$$u(x, y) = (c_1 \cos(\lambda x) + c_2 \sin(\lambda x))(c_3 \cosh(\lambda y) + c_4 \sinh(\lambda y)) =$$

$$C_1 \cos(\lambda x) \cosh(\lambda y) + C_2 \cos(\lambda x) \sinh(\lambda y) + C_3 \sin(\lambda x) \cosh(\lambda y) + C_4 \sin(\lambda x) \sinh(\lambda y) \quad .$$

Taking the **components**, we found **four** families of solutions, to serve us as possible building blocks for the final solution

$$u_1(x, y) = \cos(\lambda x) \cosh(\lambda y) \quad ,$$

$$u_2(x, y) = \cos(\lambda x) \sinh(\lambda y) \quad ,$$

$$u_3(x, y) = \sin(\lambda x) \cosh(\lambda y) \quad ,$$

$$u_4(x, y) = \sin(\lambda x) \sinh(\lambda y) \quad .$$

(In fact there are four other ones: $u_5(x, y) = \cosh(\lambda x) \cos(\lambda y)$, $u_6(x, y) = \cosh(\lambda x) \sin(\lambda y)$, $u_7(x, y) = \sinh(\lambda x) \cos(\lambda y)$, $u_8(x, y) = \sin(\lambda x) \sin(\lambda y)$, but these are out of the question for this problem).

So far, we have not looked at the **boundary conditions**. Since $u(x, 0) = 0$ but $u_1(x, 0) = \cos(\lambda x)$ is **not** zero, and $u_3(x, 0) = \sin(\lambda x)$ is **not** zero, we have to **discard** both $u_1(x, y)$ and $u_3(x, y)$. Since $u(0, y) = 0$, but $u_2(0, y) = \sinh(\lambda y)$ is **not** zero, we have to discard also $u_2(x, y)$. The only feasible option is $u_4(x, y)$ that makes the two boundary conditions $u(0, y) = 0$ and $u(x, 0) = 0$ happy. So our building blocks would be

$$u(x, y) = \sin(\lambda x) \sinh(\lambda y) \quad ,$$

where, so far, λ can be any positive real number.

Now we look at the boundary condtion:

$$u(\pi, y) = 0 \quad , \quad 0 < y < 1 \quad .$$

So

$$u(\pi, y) = \sin(\lambda\pi)\sinh(\lambda y) = 0 \quad .$$

Since $\sinh(\lambda y)$ better not be zero, this means that we have to solve the trig-equation

$$\sin(\lambda\pi) = 0 \quad .$$

Since the solution of $\sin w = 0$ is $w = n\pi$ (n integer), we have

$$\lambda\pi = n\pi \quad .$$

Dividing by π:

$$\lambda = n \quad .$$

So the only feasible building blocks are $\sin(nx)\sinh(ny)$. For any **integer** n, the pde plus the three boundary conditions

$$u(0, y) = 0 \quad , \quad u(\pi, y) = 0 \quad , \quad (0 < y < 1) \quad ;$$

$$u(x, 0) = 0 \quad , \quad (0 < x < \pi) \quad ,$$

are automatically satisfied. By the famous **superposition** principal, the same is true for any (finite or) **infinite linear combination**:

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin(nx)\sinh(ny) \quad , \qquad (Template)$$

for **any** numbers A_n, you name them! Now it is time to pick these numbers so that the last boundary condition:

$$u(x, 1) = f(x) \quad , \quad 0 < x < \pi \quad ,$$

is satisfied. Plugging-in $y = 1$ into the above template:

$$u(x, 1) = \sum_{n=1}^{\infty} A_n \sin(nx)\sinh(n) \quad ,$$

and this better be equal to $f(x)$, so we need A_n such that

$$f(x) = \sum_{n=1}^{\infty} (A_n \sinh(n))\sin(nx) \quad .$$

But this rings a bell! Recall that the coefficients, a_n in the (half-range) Fourier Sine series

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx) \quad ,$$

are given by the formula

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx \quad .$$

Equating we get

$$A_n \sinh(n) = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \quad .$$

Dividing by $\sinh n$, we get a formula for A_n:

$$A_n = \frac{2}{\pi \sinh n} \int_0^{\pi} f(x) \sin nx \, dx \quad .$$

Going back to the template we get: **Ans.**:

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin(nx) \sinh(ny) \quad ,$$

where the numbers A_n are given by: $A_n = \frac{2}{\pi \sinh n} \int_0^{\pi} f(x) \sin nx \, dx$. This is the **ans.** .

Comment: If the problem gives you a **specific** function rather than the abstract $f(x)$, for example:

$$u_{xx} + u_{yy} = 0 \quad , \quad 0 < x < \pi \quad , \quad 0 < y < 1 \quad ,$$

Subject to

$$u(0, y) = 0 \quad , \quad u(\pi, y) = 0 \quad , \quad 0 < y < 1 \quad ;$$

$$u(x, 0) = 0 \quad , \quad u(x, 1) = x \quad , \quad 0 < x < \pi \quad .$$

Then you would do as above, but at the end find the Fourier Sine expansion of the specific function (in this case x) by doing the above integration.