Solutions to the Attendance Quiz for Lecture 16

1. Solve the boundary value problem

$$u_{xx} = u_{tt} \quad , \quad 0 < x < \pi \quad , \quad t > 0 \quad ;$$

$$u(0,t) = 0 \quad , \quad u(\pi,t) = 0 \quad , \quad t > 0 \quad ;$$

$$u(x,0) = 3 \quad , \quad u_t(x,0) = 4 \quad , \quad 0 < x < \pi \quad .$$

Sol.

(Note: the most straightforward way is just to plug-in into the formula in the cheatsheet, below I present a slightly more conceptual approach, that is in tune with the "short cut" that I presented for the special case where $u(x,0)$ is a pure sine function or a combination thereof)

First note that in this problem $a = 1$. We first find the **Fourier Sine** Expansion of the function 1 over $[0,\pi]$. By the general formula

$$1 = \sum_{n=1}^{\infty} B_n \sin nx \quad ,$$

where

$$B_n = \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin nx \, dx \quad .$$

Since $f(x) = 1$, we have

$$B_n = \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin nx = \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx = \frac{2}{\pi} \cdot \frac{-\cos nx}{n} \Big|_0^{\pi}$$

$$= \frac{2}{\pi} \cdot \frac{-\cos n\pi + \cos n \cdot 0}{n} = \frac{2}{\pi} \cdot \frac{-(-1)^n + 1}{n} = \frac{2(1-(-1)^n)}{n\pi}$$

So

$$1 = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi} \sin nx \quad .$$

So immediately we get the Fourier Sine expansions of $u(x,0) = 3$ and $u_t(x,0) = 4$.

$$3 = \sum_{n=1}^{\infty} \frac{6(1-(-1)^n)}{n\pi} \sin nx \quad .$$

$$4 = \sum_{n=1}^{\infty} \frac{8(1-(-1)^n)}{n\pi} \sin nx \quad .$$

The contribution to $u(x,t)$ arising from $u(x,0)$ is gotten by simply copying-and-pasting the Fourier Sine expression above for $u(x,0)=3$ and sticking $\cos nat$ inside the summand, getting (since $a=1$) that the contribution due to $u(x,0)$ is:

$$\sum_{n=1}^{\infty} \frac{6(1-(-1)^n)}{n\pi} \sin nx \cos nt \quad .$$

The contribution to $u(x,t)$ arising from $u_t(x,0)$ is gotten by simply copying-and-pasting the Fourier Sine expression above for $u_t(x,0)=4$ and sticking $\frac{\sin nat}{na}$ inside the summand, getting (since $a=1$) that the contribution due to $u_t(x,0)$ is:

$$\sum_{n=1}^{\infty} \frac{8(1-(-1)^n)}{n\pi} \sin nx \frac{\sin nt}{n} = \sum_{n=1}^{\infty} \frac{8(1-(-1)^n)}{n^2\pi} \sin nx \sin nt \quad .$$

Finally, *adding* these two contributions yields the **answer**:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{6(1-(-1)^n)}{n\pi} \sin nx \cos nt + \sum_{n=1}^{\infty} \frac{8(1-(-1)^n)}{n^2\pi} \sin nx \sin nt \quad .$$

This would give you full credit. But since $(1-(-1)^n)=0$ when n is an even integer and it equals 2 when n is an odd integer, we can write $n=2k+1$ and get an even better answer:

$$u(x,t) = \sum_{k=0}^{\infty} \frac{12}{(2k+1)\pi} \sin(2k+1)x \cos(2k+1)t + \sum_{k=0}^{\infty} \frac{16}{(2k+1)^2\pi} \sin(2k+1)x \sin(2k+1)t \quad .$$