Solutions to the Attendance Quiz for Lecture 13

1. Find product solutions, if possible, to the partial differential equation

$$2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 0 \quad .$$

Sol.: We write

$$u(x,y) = X(x)Y(y) \quad ,$$

where $X(x)$ is a function of **only** x and $Y(y)$ is a function of **only** y. Entering it in the pde:

$$0 = 2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 2\frac{\partial (XY)}{\partial x} + 3\frac{\partial (XY)}{\partial y} = 2X_x Y + 3XY_y$$

So

$$2X'(x)Y(y) + 3X(x)Y'(y) = 0 \quad .$$

Dividing by XY:

$$\frac{2X'(x)Y(y) + 3X(x)Y'(y)}{X(x)Y(y)} = 0 \quad .$$

Algebra:

$$2\frac{X'(x)}{X(x)} + 3\frac{Y'(y)}{Y(y)} = 0 \quad .$$

So:

$$2\frac{X'(x)}{X(x)} = -3\frac{Y'(y)}{Y(y)} \quad .$$

The left side **does not** depend on y and the right side **does not** depend on x. But they are **equal**!. So *neither side* depends on x *and* y. So the are **both** equal to the **same** *constant*, let's call it k. We now have two odes:

$$2\frac{X'(x)}{X(x)} = k \quad .$$

$$-3\frac{Y'(y)}{Y(y)} = k \quad .$$

So

$$\frac{X'(x)}{X(x)} = k/2 \quad ,$$

$$\frac{Y'(y)}{Y(y)} = -k/3 \quad .$$

These are the same as

$$X'(x) - (k/2)X(x) = 0 \quad .$$

$$Y'(y) + (k/3)Y(y) = 0 \quad .$$

The **general solutions** are

$$X(x) = c_1 e^{(k/2)x} \quad ,$$

$$Y(y) = c_2 e^{-(k/3)y} \quad .$$

Going back to $u(x,y) = X(x)Y(y)$, we (almost) finally get

$$u(x,y) = c_1 c_2 e^{(k/2)x} e^{-(k/3)y} = C e^{k(x/2 - y/3)} \quad ,$$

where we write $C = c_1 c_2$ (c_1, c_2 are *arbitrary* constants, so C, their product is yet another arbitrary constant).

Ans. to 1: A product solution of the pde is $u(x,y) = Ce^{k(\frac{x}{2} - \frac{y}{3})}$, where C and k are any constants.

Comment: About %30 of the people got it completely right. A common error was $e^{k(\frac{x}{2} + \frac{y}{3})}$. Watch out for the sign, and review how to solve simple odes like these ones.

2. Check that $u_1(x,y) = 2\sin(x+y) + 3e^{x+y}$ is a solution of

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \quad .$$

Sol. Since the pde is **homogeneous** (the right side is 0), by the **superposition principle** we can check each piece **separately**.

First piece: $2\sin(x+y)$

$$\frac{\partial^2 (2\sin(x+y))}{\partial x^2} - \frac{\partial^2 (2\sin(x+y))}{\partial y^2} = (2\sin(x+y))_{xx} - (2\sin(x+y))_{yy} \quad .$$

By the chain rule $(2\sin(x+y))_x = 2\cos(x+y)$, so $(2\sin(x+y))_{xx} = (2\cos(x+y))_x = -2\sin(x+y)$. Similarly:$(2\sin(x+y))_y = 2\cos(x+y)$, so $(2\sin(x+y))_{yy} = (2\cos(x+y))_y = -2\sin(x+y)$. So:

$$\frac{\partial^2 (2\sin(x+y))}{\partial x^2} - \frac{\partial^2 (2\sin(x+y))}{\partial y^2} = -2\sin(x+y) - (-2\sin(x+y)) = 0 \quad .$$

Second piece:

$$\frac{\partial^2 (3e^{x+y})}{\partial x^2} - \frac{\partial^2 (3e^{x+y})}{\partial y^2} = (3e^{x+y})_{xx} - (3e^{x+y})_{yy} \quad .$$

By the chain rule $(3e^{x+y})_x = 3e^{x+y}$, so $(3e^{x+y})_{xx} = (3e^{x+y})_x = 3e^{x+y}$, Similarly: $(3e^{x+y})_y = 3e^{x+y}$, so $(3e^{x+y})_{yy} = (3e^{x+y})_y = 3e^{x+y}$, So:

$$\frac{\partial^2 (3e^{x+y})}{\partial x^2} - \frac{\partial^2 (3e^{x+y}))}{\partial y^2} = 3e^{x+y} - 3e^{x+y} = 0 \quad .$$

It follows that $u_1(x,y) = 2\sin(x+y) + 3e^{x+y}$, the sum of the two pieces is also a solution of the given pde.

Comment: Most people (who had time) got it right.