Solution to the Attendance Quiz for Lecture 12

1. Find the first three coefficients of the Fourier-Legendre expansion of

$$f(x) = \begin{cases} 2, & \text{if } -1 < x < 0; \\ 3, & \text{if } 0 \le x < 1. \end{cases}$$

Sol. Recall that

$$P_0(x) = 1 \quad , P_1(x) = 1 \quad , \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} \quad .$$

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx \quad .$$

$$c_0 = \frac{2 \cdot 0 + 1}{2} \int_{-1}^{1} f(x) P_0(x)\, dx = \frac{1}{2} \int_{-1}^{1} f(x)\, dx = \frac{1}{2} \int_{-1}^{0} f(x)\, dx + \frac{1}{2} \int_{0}^{1} f(x)\, dx$$

$$\frac{1}{2} \int_{-1}^{0} (2)\, dx + \frac{1}{2} \int_{0}^{1} (3)\, dx = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 3 = \frac{5}{2} \quad .$$

$$c_1 = \frac{2 \cdot 1 + 1}{2} \int_{-1}^{1} f(x) P_1(x)\, dx = \frac{3}{2} \int_{-1}^{1} f(x)x, dx = \frac{3}{2} \int_{-1}^{0} f(x)x\, dx + \frac{3}{2} \int_{0}^{1} f(x)x\, dx$$

$$\frac{3}{2} \int_{-1}^{0} 2x\, dx + \frac{3}{2} \int_{0}^{1} 3x\, dx = \frac{3}{2}(x^2)\Big|_{-1}^{0} + \frac{3}{2}\frac{3x^2}{2}\Big|_{0}^{1}$$

$$= \frac{3}{2}(0^2 - (-1)^2) + \frac{9}{4}(1^2 - 0^2) = -\frac{3}{2} + \frac{9}{4} = \frac{3}{4}$$

$$c_2 = \frac{2 \cdot 2 + 1}{2} \int_{-1}^{1} f(x) P_2(x)\, dx = \frac{5}{2} \int_{-1}^{1} f(x)(\frac{3}{2}x^2 - \frac{1}{2})\, dx = \frac{5}{2} \int_{-1}^{0} f(x)(\frac{3}{2}x^2 - \frac{1}{2})\, dx + \frac{5}{2} \int_{0}^{1} f(x)(\frac{3}{2}x^2 - \frac{1}{2})\, dx \quad .$$

$$= \frac{5}{2} \int_{-1}^{0} 2(\frac{3}{2}x^2 - \frac{1}{2})\, dx + \frac{5}{2} \int_{0}^{1} 3(\frac{3}{2}x^2 - \frac{1}{2})\, dx \quad .$$

$$= \frac{5}{2} \int_{-1}^{0} (3x^2 - 1)\, dx + \frac{5}{2} \int_{0}^{1} (\frac{9}{2}x^2 - \frac{3}{2})\, dx \quad .$$

$$= \frac{5}{2}(x^3 - x)\Big|_{-1}^{0} + (\frac{45}{4}\frac{x^3}{3} - \frac{15}{4}x)\Big|_{0}^{1}$$

$$= \frac{5}{2}(-1 - (-1) - 0) + (\frac{45}{4}\frac{1^3 - 0^3}{3} - \frac{15}{4}(1 - 0) = 0 \quad .$$

Ans. the first three coefficients of the Fourier-Legendre expansion of $f(x)$ are $c_0 = \frac{5}{2}, c_1 = \frac{3}{4}, c_2 = 0$.

Comment: Only about %30 of the people got it completely correct, but many got c_0 and c_1 and most people did it the right way.

Common mistake: **Many** people wrote

$$f(x) = \frac{5}{2}P_0(x) + \frac{3}{4}P_1(x) + 0 \cdot P_2(x)$$

This is **WRONG**. If you want to write it in this format, you **MUST** write

$$f(x) = \frac{5}{2}P_0(x) + \frac{3}{4}P_1(x) + 0 \cdot P_2(x) + \ldots \quad .$$

The ... are **crucial**! It means that the Fourier-Legendre series goes **for ever**, and what you got is the very start of an infinite journey.

Of course it is wrong that $f(x) = \frac{5}{2}P_0(x) + \frac{3}{4}P_1(x) + 0 \cdot P_2(x)$ (withut the ...). The right side is $y = \frac{5}{2} + \frac{3}{4}x$ a **straight line** and **not** the discontinuous function of the problem.